Polyhedral Compilation without Polyhedra

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Outline

Part I

Polyhedral Compilation

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Analyzing and/or transforming loop programs using the *polyhedral model*

Polyhedral Model

Abstract representation of a loop program

- instance based
  - ⇒ statement *instances*
  - ⇒ array *elements*
- compact representation based on polyhedra or similar objects
  - ⇒ integer points in unions of parametric polyhedra
  - ⇒ Presburger sets and relations
- parametric
  - ⇒ description may depend on symbolic constants

[8, 10, 14, 16]

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Note: naming is "historical"

- polyhedral compilation does not require polyhedra (e.g., `omega(+)`)
- other approaches also use polyhedra (e.g., abstract interpretation)

Polyhedral Model

Main constituents of program representation

- Instance Set
  - $\Rightarrow$ the set of all statement instances
- Access Relations
  - $\Rightarrow$ the array elements accessed by a statement instance
- Dependences
  - $\Rightarrow$ the statement instances that depend on a statement instance
- Schedule
  - $\Rightarrow$ the relative execution order of statement instances

Polyhedral Model Requirements

Requirements for basic polyhedral model: SANA input

- Static control
  $\Rightarrow$ control does not depend on input data
- Affine
  $\Rightarrow$ all relevant expressions are (quasi-)affine
- No Aliasing
  $\Rightarrow$ essentially no pointer manipulations

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Note:

- polyhedral model may be *approximation* of input that does not strictly satisfy all requirements
- many *extensions* are available
  a small selection of these extensions will be discussed in this tutorial

Illustrative Example

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R:   h(A[2]);
     for (int i = 0; i < 2; ++i)
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- instance based

- Instance Set (set of statement instances)

$$I = \{\mathtt{R}(); \mathtt{S}(0,0); \mathtt{S}(0,1); \mathtt{S}(1,0); \mathtt{S}(1,1); \mathtt{T}(0); \mathtt{T}(1)\}$$

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- Access Relations (accessed array elements: Writes W, Reads R)

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Constraints

- "read off", or
- obtained through abstract interpretation

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$$R = \{\{\mathtt{S}(i,j) \to \mathtt{A}(i+j) : 0 \le i < 2 \wedge 0 \le j < 2\} \to \mathtt{A}(0)\}$$

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Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
      for (int k = 0; k < K; k++)
S2:       C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
```

- Instance Set (set of statement instances)

$$\{\, \mathtt{S1}(i,j) : 0 \le i < M \wedge 0 \le j < N;$$
$$\mathtt{S2}(i,j,k) : 0 \le i < M \wedge 0 \le j < N \wedge 0 \le k < K \,\}$$

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$$R = \{\, \mathtt{S2}(i,j,k) \to \mathtt{C}(i,j); \mathtt{S2}(i,j,k) \to \mathtt{A}(i,k); \mathtt{S2}(i,j,k) \to \mathtt{B}(k,j) \,\}$$

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Named Presburger Sets and Relations

[20]

Examples

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General form

- Sets
$$\{\, S_1(\mathbf{i}) : f_1(\mathbf{i}); S_2(\mathbf{i}) : f_2(\mathbf{i}); \ldots \},$$
with f_k Presburger formulas
$\Rightarrow$ set of elements of the form $S_1(\mathbf{i})$, one for each $\mathbf{i}$ satisfying $f_1(\mathbf{i})$, ...
- Binary relations
$$\{\, S_1(\mathbf{i}) \to T_1(\mathbf{j}) : f_1(\mathbf{i},\mathbf{j}); S_2(\mathbf{i}) \to T_2(\mathbf{j}) : f_2(\mathbf{i},\mathbf{j}); \ldots \}$$
$\Rightarrow$ set of pairs of elements of the form $S_1(\mathbf{i}) \to T_1(\mathbf{j})$

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Note: despite "$\to$", not necessarily (single valued) functions

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Note: may depend on interpretation of symbolic constants

$$\{ \mathrm{S}(i) : 0 \leq i \leq \mathrm{n}() \}$$

is equal to

$$\begin{cases} \emptyset & \text{if } \mathrm{n} < 0 \\ \{ \mathrm{S}(0) \} & \text{if } \mathrm{n} = 0 \\ \{ \mathrm{S}(0); \mathrm{S}(1) \} & \text{if } \mathrm{n} = 1 \\ \{ \mathrm{S}(0); \mathrm{S}(1); \mathrm{S}(2) \} & \text{if } \mathrm{n} = 2 \\ \dots \end{cases}$$

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- Presburger Formula
  - true
  - equality on terms $(=)$
  - less than or equal on terms $(\leq)$
  - logical connectives $(\wedge, \vee, \neg)$
  - quantification $(\exists, \forall)$

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  Example: $\{ \mathrm{S}(i) \rightarrow \mathrm{S}(i+1) \}$ is equal to $\{ \mathrm{S}(i) \rightarrow \mathrm{S}(j) : j = i+1 \}$

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- $a < b$ is equal to $a \leq b - 1$
- $a \geq b$ is equal to $b \leq a$
- $a > b$ is equal to $a \geq b + 1$

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- $\{ S(i_1, \ldots i_{n-1}, g(i_1, \ldots, i_{n-1}), i_{n+1}, \ldots) : f(\mathbf{i}) \}$ is equal to $\{ S(i_1, \ldots i_{n-1}, i_n, i_{n+1}, \ldots) : f(\mathbf{i}) \wedge i_n = g(i_1, \ldots, i_{n-1}) \}$
  Example: $\{ \mathrm{S}(i) \rightarrow \mathrm{S}(i+1) \}$ is equal to $\{ \mathrm{S}(i) \rightarrow \mathrm{S}(j) : j = i+1 \}$
- n is equal to $n()$
- $a < b$ is equal to $a \leq b - 1$
- $a \geq b$ is equal to $b \leq a$
- $a > b$ is equal to $a \geq b + 1$
- $a, b \oplus c$ is equal to $a \oplus c \wedge b \oplus c$ with $\oplus \in \{ \leq, <, \geq, >, = \}$
  Example: $\{ \mathrm{S}(i,j) : i,j \geq 0 \}$ is equal to $\{ \mathrm{S}(i,j) : i \geq 0 \wedge j \geq 0 \}$

Syntactic Sugar

- false is equal to $\neg\mathrm{true}$
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- $a \oplus_1 b \oplus_2 c$ is equal to $a \oplus_1 b \wedge b \oplus_2 c$ with $\{ \oplus_1, \oplus_2 \} \subset \{ \leq, <, \geq, >, = \}$
  Example: $\{ \mathrm{S}(i) : 0 \leq i \leq 10 \}$ is equal to $\{ \mathrm{S}(i) : 0 \leq i \wedge i \leq 10 \}$

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- $-e$ is equal to $0 - e$

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- $a_1, \ldots, a_n \succcurlyeq b_1, \ldots, b_n$ is equal to $b_1, \ldots, b_n \preccurlyeq a_1, \ldots, a_n$
- a mod n is equal to $a - n \lfloor a/n \rfloor$

Spaces

Recall general form

- Sets

$$\{ S_1(\mathbf{i}) : f_1(\mathbf{i}); S_2(\mathbf{i}) : f_2(\mathbf{i}); \dots \},$$

- Binary relations

$$\{ S_1(\mathbf{i}) \rightarrow T_1(\mathbf{j}) : f_1(\mathbf{i},\mathbf{j}); S_2(\mathbf{i}) \rightarrow T_2(\mathbf{j}) : f_2(\mathbf{i},\mathbf{j}); \dots \}$$

The identifier (e.g., S_1, S_2, T_1, T_2), together with
the dimension, i.e., number of elements in subsequent tuple (e.g., $\mathbf{i}$, $\mathbf{j}$),
will be called a space

When we say $S_2(\mathbf{i}) = T_1(\mathbf{j})$, we mean

- the identifiers S_2 and T_1 are the same
- the dimensions of $\mathbf{i}$ and $\mathbf{j}$ are the same

Examples: $\mathrm{S}() \neq \mathrm{S}(i)$, $\mathrm{S}(a) = \mathrm{S}(b)$, $\mathrm{S}() \neq \mathrm{T}()$

Space Decomposition

General form can be rewritten

$$\{ S_1(\mathbf{i}) \to T_1(\mathbf{j}) : f_1(\mathbf{i},\mathbf{j}); S_2(\mathbf{i}) \to T_2(\mathbf{j}) : f_2(\mathbf{i},\mathbf{j}); \dots \}$$
$$= \bigcup_k \{ S_k(\mathbf{i}) \to T_k(\mathbf{j}) : f_k(\mathbf{i},\mathbf{j}) \}$$

- some operations distribute with union
- other operations are defined on a single (pair of) space(s)
  - $\Rightarrow$ "space local" operations
  - $\Rightarrow$ replace $\{ S(\mathbf{i}) \to T(\mathbf{j}) : f_1(\mathbf{i},\mathbf{j}); S(\mathbf{i}) \to T(\mathbf{j}) : f_2(\mathbf{i},\mathbf{j}) \}$ by $\{ S(\mathbf{i}) \to T(\mathbf{j}) : f_1(\mathbf{i},\mathbf{j}) \vee f_2(\mathbf{i},\mathbf{j}) \}$

In both cases, we define

- unary operator $\oplus$

$$\oplus \bigcup_i R_i := \bigcup_i \oplus R_i$$

- binary operator $\oplus$

$$\left(\bigcup_i R_i \right) \oplus \left(\bigcup_j S_j \right) := \bigcup_i \bigcup_j (R_i \oplus S_j)$$

Basic Operations

$$A = \{\, S(\mathbf{i}_1) \rightarrow T(\mathbf{j}_1) : f(\mathbf{i}_1, \mathbf{j}_1) \,\} \quad B = \{\, U(\mathbf{i}_2) \rightarrow V(\mathbf{j}_2) : g(\mathbf{i}_2, \mathbf{j}_2) \,\}$$

- Union

  $$A \cup B = \{\, S(\mathbf{i}) \rightarrow T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}); U(\mathbf{i}) \rightarrow V(\mathbf{j}) : g(\mathbf{i}, \mathbf{j}) \,\}$$

- Intersection

  $$A \cap B = \begin{cases} \{\, S(\mathbf{i}) \rightarrow T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}) \wedge g(\mathbf{i}, \mathbf{j}) \,\} & \text{if } S(\mathbf{i}_1) = U(\mathbf{i}_2) \text{ and } T(\mathbf{j}_1) = V(\mathbf{j}_2) \\ \emptyset & \text{otherwise} \end{cases}$$

- Difference

  $$A \setminus B = \begin{cases} \{\, S(\mathbf{i}) \rightarrow T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}) \wedge \neg g(\mathbf{i}, \mathbf{j}) \,\} & \text{if } S(\mathbf{i}_1) = U(\mathbf{i}_2) \text{ and } T(\mathbf{j}_1) = V(\mathbf{j}_2) \\ \{\, S(\mathbf{i}) \rightarrow T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}) \,\} & \text{otherwise} \end{cases}$$

Emptiness Check and Comparisons

- Emptiness check
  Does a set or binary relation contain any elements?
  (for any value of the symbolic constants)

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- Comparisons
  - $A \subseteq B$ is defined as $A \setminus B = \emptyset$
  - $A \supseteq B$ is defined as $B \subseteq A$
  - $A = B$ is defined as $A \subseteq B \wedge A \supseteq B$
  - $A \subset B$ is defined as $A \subseteq B \wedge \neg(A = B)$
  - $A \supset B$ is defined as $B \subset A$

Cardinality

- Cardinality of a set
  - $\Rightarrow$ number of elements in the set
  - $\Rightarrow$ may depend on symbolic constants

$S = \{ S(\mathbf{i}) : f(\mathbf{i}) \}$

$\text{card}\, S = \{ n : n = \#\mathbf{i} : f(\mathbf{i}) \}$

$\text{card}\, \{ \mathtt{A}(i) : 0 \leq i \leq n; \mathtt{B}() \} = n + 2$

$$\text{card} \left(\bigcup_i S_i \right) := \sum_i \text{card}\, S_i$$

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- Cardinality of a binary relation
  - $\Rightarrow$ for each domain element, number of corresponding images

$R = \{ S(\mathbf{i}) \to T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}) \}$

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$R = \{ \mathtt{A}(i) \to \mathtt{C}(i) : 0 \leq i \leq n; \mathtt{B}() \to \mathtt{C}(i) : 0 \leq i \leq n \}$
$\operatorname{card} R = \{ \mathtt{A}(i) \to 1 : 0 \leq i \leq n; \mathtt{B}() \to n + 1 \}$

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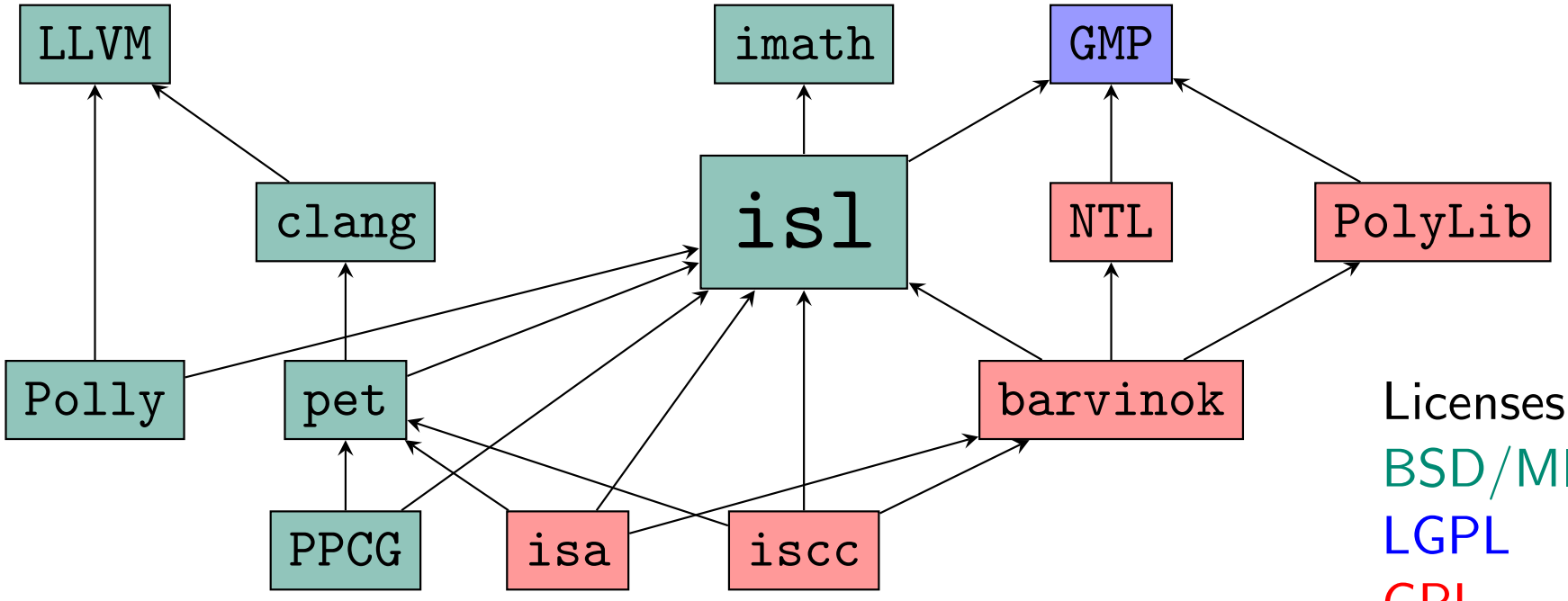
$\Rightarrow$ not a Presburger formula

isl and Related Libraries and Tools



isl: manipulates parametric affine sets and relations

barvinok: counts elements in parametric affine sets and relations

pet: extracts polyhedral model from clang AST

PPCG: Polyhedral Parallel Code Generator

iscc: interactive calculator

isa: prototype tool set including derivation of process networks and equivalence checker

isl/iscc syntax

Relation description

() (tuple)	`[]`
$+, -$	`+, -`
$=, \leq, <, \geq, >$	`=, <=, <, >=, >`
true	`true`
false	`false`
$\wedge$	`and`
$\vee$	`or`
$\neg$	`not`
$\exists v :$	`exists v :`
$\forall v :$	`not exists v : not`
$\preccurlyeq, \prec, \succcurlyeq, \succ$	(not available yet; write out explicitly)

Operations on relations

$\cup$	`+`
$\cap$	`*`
$\setminus$	`-`
$=$	`=`
$\subset, \subseteq, \supset, \supseteq$	`<, <=, >, >=`
card	`card`

Note: symbolic constants need to be explicitly declared

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
      for (int k = 0; k < K; k++)
S2:       C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
```

- Number of statement instances

$$\text{card}\{\, \mathtt{S1}(i,j) : 0 \le i < M \wedge 0 \le j < N;$$
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- Number of array elements accessed by each instance

$$\text{card}\{\, \mathtt{S1}(i,j) \to \mathtt{C}(i,j); \mathtt{S2}(i,j,k) \to \mathtt{C}(i,j);$$
$$\mathtt{S2}(i,j,k) \to \mathtt{C}(i,j); \mathtt{S2}(i,j,k) \to \mathtt{A}(i,k); \mathtt{S2}(i,j,k) \to \mathtt{B}(k,j) \,\}$$

Exercise

```
int f1(int m, int n, int A[const static m][n])
{
        int t = 0;
        for (int i = 0; i < m; ++i)
                t += A[i][i];
        return t;
}
void f2(int m, int n, int A[/*.*/m][n][n], int B[/*.*/m][n])
{
        for (int i = 0; i < m; ++i) {
S:              B[i][0] = 0;
                for (int j = 0; j < n; ++j) {
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How many statement instances are executed by `f2`?

How many array elements accessed by each instance?

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```

How many statement instances are executed by f2? `m>n ? m*n-n+m : m*n`

How many array elements accessed by each instance? `S[i]->1;T[i,j]->1+j`

Outline

Polyhedral Transformation

Two approaches

1. encode execution order in statement instance indices
   $\Rightarrow$ transformation performed by manipulating instance set
2. keep track of execution order separately: schedule
   $\Rightarrow$ transformation performed by manipulating initial/current schedule, or
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Schedule O keeps track of relative execution order of statement instances

$\Rightarrow$ for each pair of statement instances $S(\mathbf{i})$ and $T(\mathbf{j})$, schedule determines

- $S(\mathbf{i})$ executed before $T(\mathbf{j})$ $\quad O(S(\mathbf{i}), T(\mathbf{j})) < 0$
- $S(\mathbf{i})$ executed after $T(\mathbf{j})$ $\quad O(S(\mathbf{i}), T(\mathbf{j})) > 0$
- $S(\mathbf{i})$ and $T(\mathbf{j})$ may be executed simultaneously $\quad O(S(\mathbf{i}), T(\mathbf{j})) = 0$

Schedule Representations

Types of schedule representations

- Combined representations
  - schedule tree
  - named Presburger relation
- Scattered representations
  - Kelly's abstraction
  - "$2d + 1$"

[11, 13, 20, 24]

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Schedule Trees

- Main node types
  - sequence: children are executed in order
  - band: instance are executed according to associated piecewise quasi-affine partial schedule P
- Deriving schedule tree from AST
  - `for` loop $\Rightarrow$ single-dimensional band
  - compound statement $\Rightarrow$ sequence

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    }
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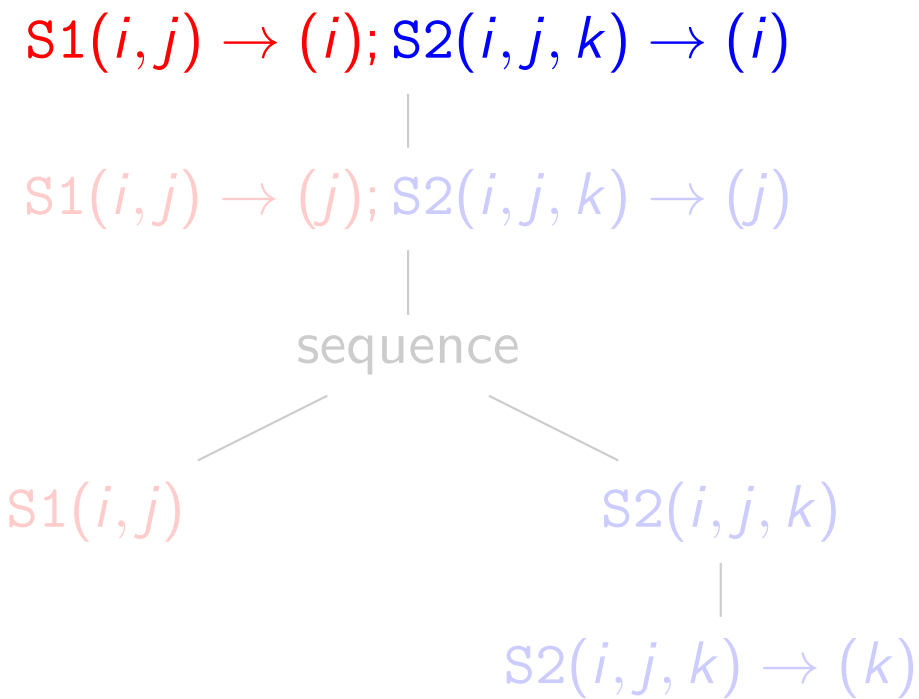

Parametric Example: Matrix Multiplication

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for (int i = 0; i < M; i++)
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    }
```

$$\mathtt{S1}(i,j) \to (i); \mathtt{S2}(i,j,k) \to (i)$$

$$\mathtt{S1}(i,j) \to (j); \mathtt{S2}(i,j,k) \to (j)$$

sequence

$\mathtt{S1}(i,j)$

$\mathtt{S2}(i,j,k)$

$\mathtt{S2}(i,j,k) \to (k)$

Parametric Example: Matrix Multiplication

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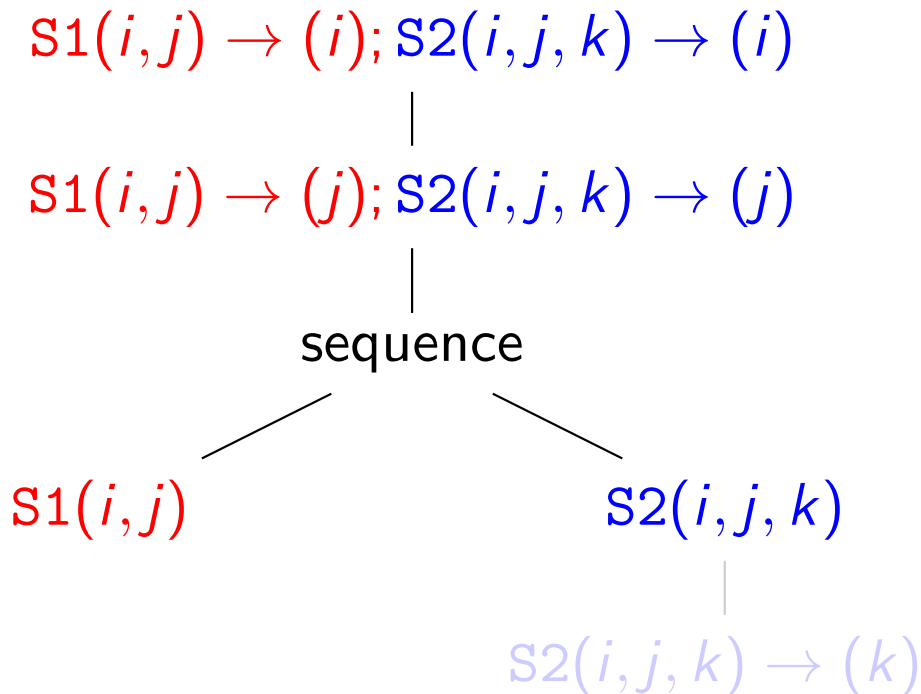

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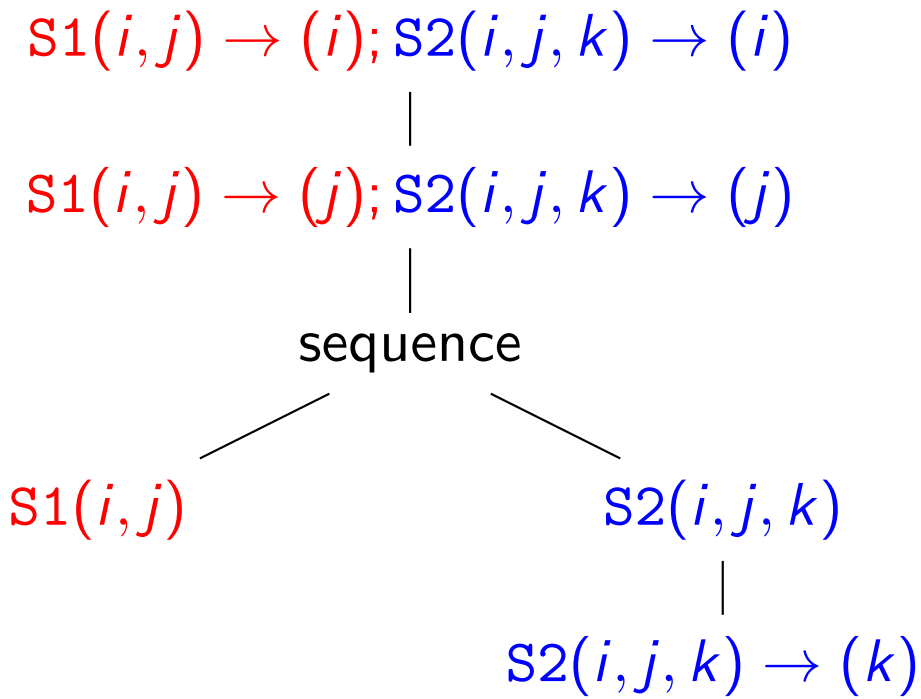


Schedule Trees — Execution Order

What is the execution order of statement instances $S(\mathbf{i})$ and $T(\mathbf{j})$ determined by schedule tree O?

Start at root of schedule tree
while *current node is not a leaf* **do**
  if *current node is sequence* **then**
    if $S(\mathbf{i})$ *and* $T(\mathbf{j})$ *appear in same child* **then**
      Move to common child
    else if $S(\mathbf{i})$ *appears in earlier child* **then**
      return $O(S(\mathbf{i}), T(\mathbf{j})) < 0$
    else
      return $O(S(\mathbf{i}), T(\mathbf{j})) > 0$
  else
    if $P(S(\mathbf{i})) = P(T(\mathbf{j}))$ **then**
      Move to single child
    else if $P(S(\mathbf{i})) \prec P(T(\mathbf{j}))$ **then**
      return $O(S(\mathbf{i}), T(\mathbf{j})) < 0$
    else
      return $O(S(\mathbf{i}), T(\mathbf{j})) > 0$
return $O(S(\mathbf{i}), T(\mathbf{j})) = 0$

Named Presburger Relation Schedules

Schedule tree with single (band) node

Named Presburger Relation Schedules

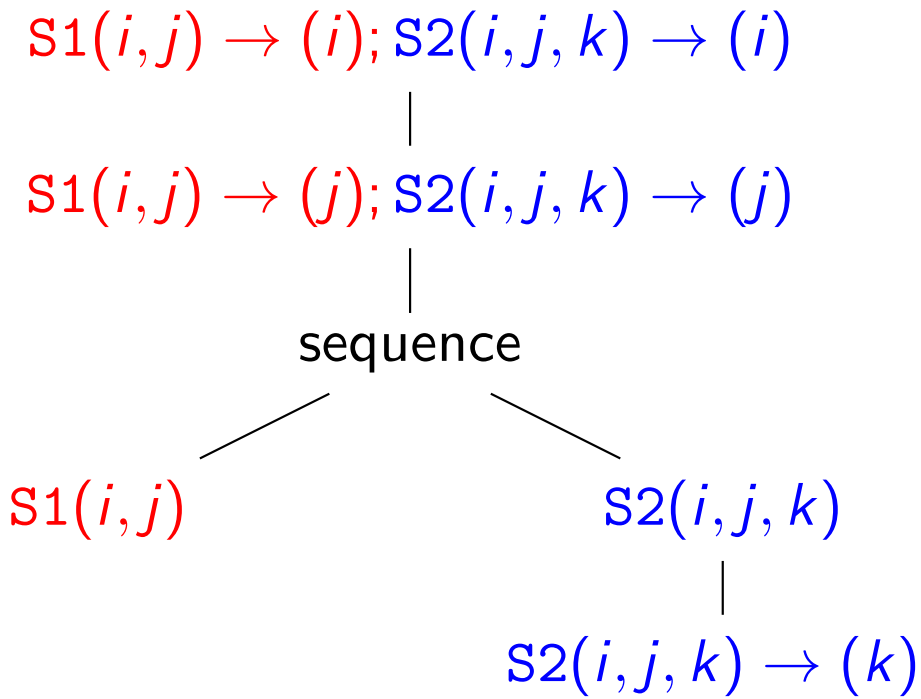
Schedule tree with single (band) node

Flattening a schedule tree

- two nested band nodes
  - $\Rightarrow$ replace by single band node with concatenated partial schedule
- sequence with as children either leaves or trees consisting of a single band node
  - $\Rightarrow$ treat leaves as zero-dimensional band nodes
  - $\Rightarrow$ pad lower-dimensional bands (e.g., with zero)
  - $\Rightarrow$ construct one-dimensional band assigning increasing value to children
  - $\Rightarrow$ combine one-dimensional band with children

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
      for (int k = 0; k < K; k++)
S2:       C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
```



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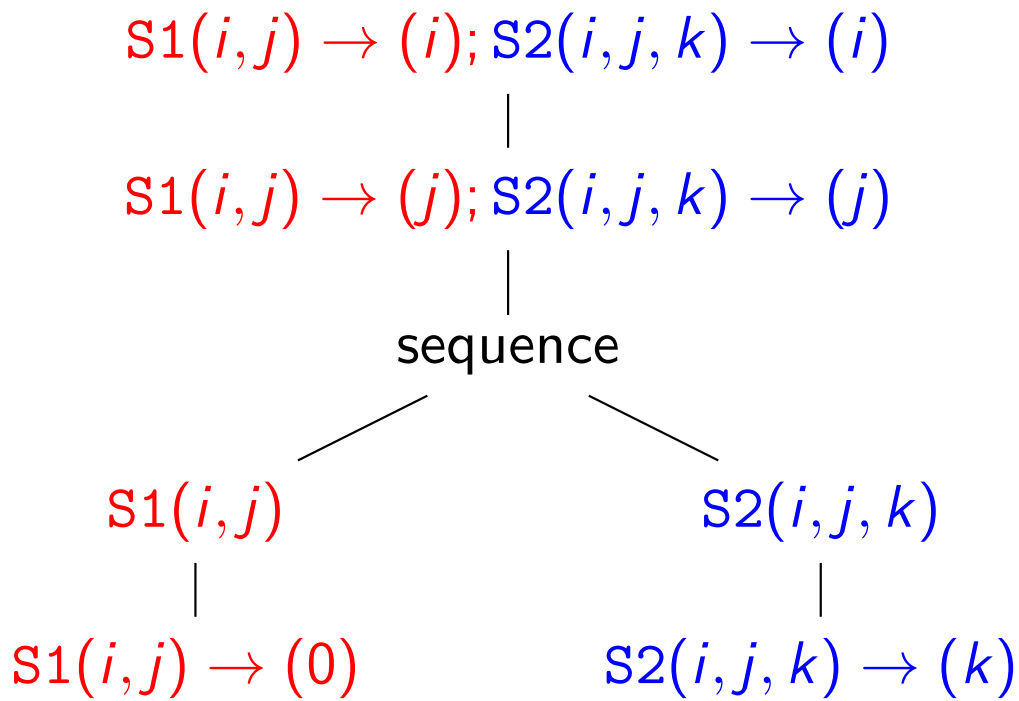

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```

$$\begin{array}{c} \mathtt{S1}(i,j) \rightarrow (i); \mathtt{S2}(i,j,k) \rightarrow (i) \\ | \\ \mathtt{S1}(i,j) \rightarrow (j); \mathtt{S2}(i,j,k) \rightarrow (j) \\ | \\ \mathtt{S1}(i,j) \rightarrow (0,0); \mathtt{S2}(i,j,k) \rightarrow (1,k) \end{array}$$

Parametric Example: Matrix Multiplication

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    }
```

$$\mathtt{S1}(i,j) \to (i); \mathtt{S2}(i,j,k) \to (i)$$

$$\Big\downarrow$$

$$\mathtt{S1}(i,j) \to (j,0,0); \mathtt{S2}(i,j,k) \to (j,1,k)$$

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    }
```

$$\mathtt{S1}(i,j) \rightarrow (i,j,0,0); \mathtt{S2}(i,j,k) \rightarrow (i,j,1,k)$$

Domain and Range of a Relation

$$R = \{ S(\mathbf{i}) \to T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}) \}$$

- Domain (`iscc: dom`)

  $\mathrm{dom}\, R = \{ S(\mathbf{i}) : \exists \mathbf{j} : f(\mathbf{i}, \mathbf{j}) \}$

  $W = \{ \mathtt{S}(i,j) \to \mathtt{A}(i+j) : 0 \le i < 2 \wedge 0 \le j < 2 \}$
  $\mathrm{dom}\, W = \{ \mathtt{S}(i,j) : 0 \le i < 2 \wedge 0 \le j < 2 \}$

  $\Rightarrow$ statement instances writing any array element

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  $\Rightarrow$ statement instances writing any array element

- Range (`iscc: ran`)

  $\mathrm{ran}\, R = \{ T(\mathbf{j}) : \exists \mathbf{i} : f(\mathbf{i}, \mathbf{j}) \}$

  $W = \{ \mathtt{S}(i,j) \to \mathtt{A}(i+j) : 0 \le i < 2 \wedge 0 \le j < 2 \}$
  $\mathrm{ran}\, W = \{ \mathtt{A}(a) : 0 \le a \le 2 \}$

  $\Rightarrow$ written array elements

Domain/Range Restriction

$$A = \{ S_1(\mathbf{i}_1) : f(\mathbf{i}_1) \} \qquad B = \{ T_1(\mathbf{j}_1) : g(\mathbf{j}_1) \}$$
$$C = \{ S_2(\mathbf{i}_2) \to T_2(\mathbf{j}_2) : h(\mathbf{i}_2, \mathbf{j}_2) \}$$

- Product relation (`iscc: ->`)

  $A \to B = \{ S_1(\mathbf{i}) \to T_1(\mathbf{j}) : f(\mathbf{i}) \wedge g(\mathbf{j}) \}$

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- Domain restriction (`iscc: *`)

$$R \cap_{dom} S = R \cap (S \to (\mathrm{ran}\, R))$$

$$C \cap_{dom} A = \begin{cases} \{ S_2(\mathbf{i}) \to T_2(\mathbf{j}) : f(\mathbf{i}) \wedge h(\mathbf{i}, \mathbf{j}) \} & \text{if } S_1(\mathbf{i}_1) = S_2(\mathbf{i}_2) \\ \emptyset & \text{otherwise} \end{cases}$$

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- Range restriction

$$R \cap_{ran} S = R \cap ((\mathrm{dom}\, R) \to S)$$

$$C \cap_{ran} A = \begin{cases} \{ S_2(\mathbf{i}) \to T_2(\mathbf{j}) : f(\mathbf{j}) \wedge h(\mathbf{i}, \mathbf{j}) \} & \text{if } S_1(\mathbf{i}_1) = T_2(\mathbf{j}_2) \\ \emptyset & \text{otherwise} \end{cases}$$

AST Generation

[5, 6]

Input:
- instance set
- schedule

Output:
- AST that visits each domain element according to the order specified by the schedule

Note: in case of flat schedule, schedule order is lexicographic order of output space

$\Rightarrow$ single output space

`iscc codegen` operation takes as input flat schedule with instance set encoded in domain

$\Rightarrow$ apply $*$ to "pure" schedule and instance set first

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
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```
I := [M,N,K] -> { S1[i,j] : 0 <= i < M and 0 <= j < N;
  S2[i,j,k] : 0 <= i < M and 0 <= j < N and 0 <= k < K };
O := { S1[i,j] -> [i,j,0,0]; S2[i,j,k] -> [i,j,1,k] };
codegen (O * I);
```

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codegen (O * I);
```

```
for (int c0 = 0; c0 < M; c0 += 1)
  for (int c1 = 0; c1 < N; c1 += 1) {
    S1(c0, c1);
    for (int c3 = 0; c3 < K; c3 += 1)
      S2(c0, c1, c3);
  }
```

Exercise

```
int f1(int m, int n, int A[const static m][n])
{
        int t = 0;
        for (int i = 0; i < m; ++i)
                t += A[i][i];
        return t;
}
void f2(int m, int n, int A[/*.*/m][n][n], int B[/*.*/m][n])
{
        for (int i = 0; i < m; ++i) {
S:              B[i][0] = 0;
                for (int j = 0; j < n; ++j) {
                        if (j == i)
                                continue;
T:                      B[i][j] = f1(j, n, A[i]);
                }
        }
}
```

Write down schedule and generate AST codegen (0 * I)

Outline

Schedule Validity

[1]

Not all schedules correspond to a valid execution order

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R(a) W(a) $\longrightarrow$ R(a) W(b) W(a) W(a)

Internal restrictions

- A read of a value should not be scheduled after the write of the value
- No other write to same memory location may be scheduled in between

External restrictions (on non-temporaries)

- No write may be scheduled before initial read from a memory location
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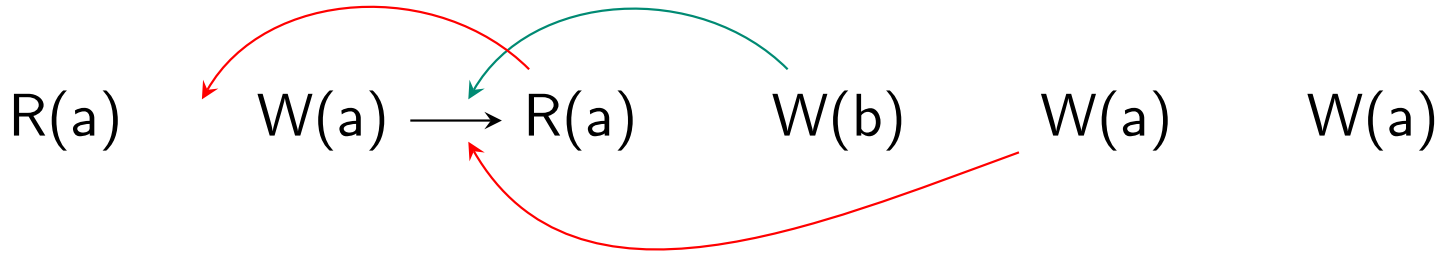

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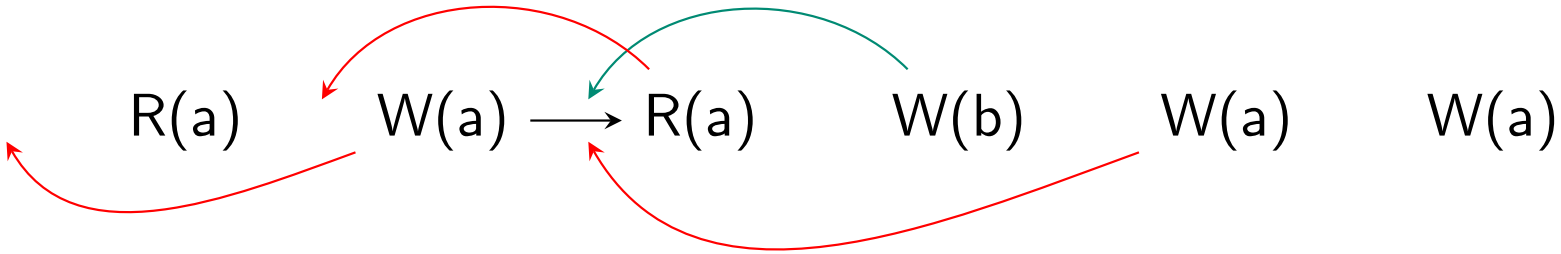

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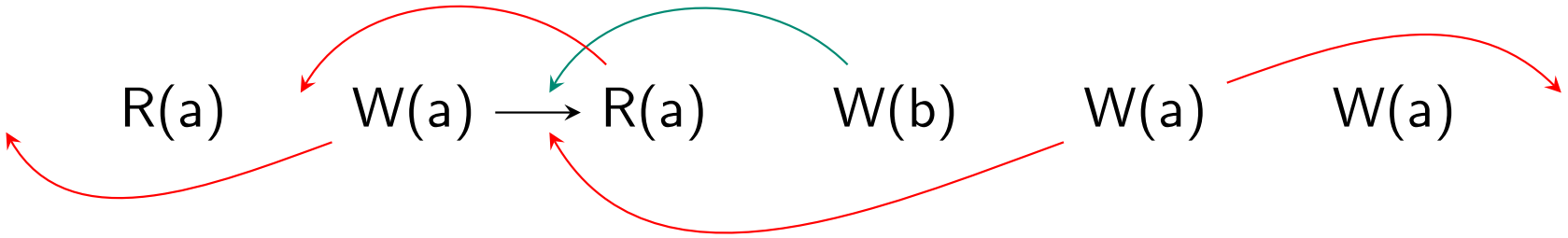

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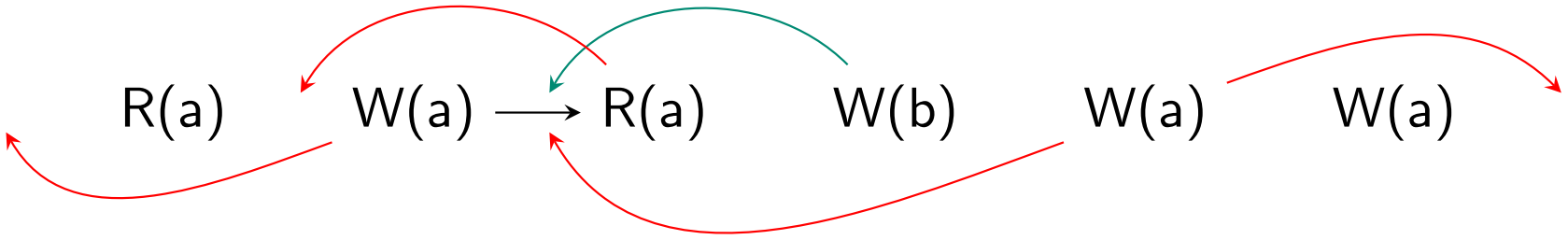

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Sufficient conditions:

- Every read of a memory location is scheduled after every previous write to the same memory location
- Every write to a memory location is scheduled after every previous read or write to the same memory location

Dependences

Sufficient conditions for schedule validity:

- Every read of a memory location is scheduled after every previous write to the same memory location
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Dependence relation D: pairs of statement instances

- accessing the same memory location
- of which at least one is a write
- with the first executed before the second

Sufficient condition:

$$\forall S(\mathbf{i}) \rightarrow T(\mathbf{j}) \in D : O(S(\mathbf{i}), T(\mathbf{j})) < 0$$

Inverse Relation and Composition

$$A = \{ S_1(\mathbf{i}_1) \to T_1(\mathbf{j}_1) : f(\mathbf{i}_1, \mathbf{j}_1) \} \qquad B = \{ S_2(\mathbf{i}_2) \to T_2(\mathbf{j}_2) : g(\mathbf{i}_2, \mathbf{j}_2) \}$$

- Inverse (`iscc: ^-1`)

  $A^{-1} = \{ T_1(\mathbf{j}) \to S_1(\mathbf{i}) : f(\mathbf{i}, \mathbf{j}) \}$

  $W = \{ \mathtt{S}(i,j) \to \mathtt{A}(i+j) : 0 \le i < 2 \wedge 0 \le j < 2 \}$
  $W^{-1} = \{ \mathtt{A}(a) \to \mathtt{S}(i,j) : a = i + j \wedge 0 \le i < 2 \wedge 0 \le j < 2 \}$

  $\Rightarrow$ statement instances writing array element

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  $\Rightarrow$ statement instances writing array element

- Composition (`iscc: after`)

$$B \circ A = \begin{cases} \{ S_1(\mathbf{i}) \to T_2(\mathbf{j}) : \exists \mathbf{k} : f(\mathbf{i}, \mathbf{k}) \wedge g(\mathbf{k}, \mathbf{j}) \} & \text{if } T_1(\mathbf{j}_1) = S_2(\mathbf{i}_2) \\ \emptyset & \text{otherwise} \end{cases}$$

  $W^{-1} \circ W = \{ \mathtt{S}(i,j) \to \mathtt{S}(i',j') : 0 \le i, i', j, j' < 2 \wedge i + j = i' + j' \}$

  $\Rightarrow$ pairs of statement instances that write same array element

Lexicographic Order

- Sets (`iscc: <<`)

$$A = \{\, S(\mathbf{i}) : f(\mathbf{i}) \,\}$$

$$B = \{\, T(\mathbf{j}) : g(\mathbf{j}) \,\}$$

$$A \prec B = \begin{cases} \{\, S(\mathbf{i}) \rightarrow S(\mathbf{j}) : f(\mathbf{i}) \wedge g(\mathbf{j}) \wedge \mathbf{i} \prec \mathbf{j} \,\} & \text{if } S(\mathbf{i}) = T(\mathbf{j}) \\ \emptyset & \text{otherwise} \end{cases}$$

Lexicographic Order

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- Relations (`iscc: <<`)

$\Rightarrow$ binary relation on domains reflecting lexicographic order of images

$$A = \{ S_1(\mathbf{i}_1) \to T_1(\mathbf{j}_1) : f(\mathbf{i}_1, \mathbf{j}_1) \}$$
$$B = \{ S_2(\mathbf{i}_2) \to T_2(\mathbf{j}_2) : g(\mathbf{i}_2, \mathbf{j}_2) \}$$
$$A \prec B = \begin{cases} \{ S_1(\mathbf{i}_1) \to S_2(\mathbf{i}_2) : \exists \mathbf{j}_1, \mathbf{j}_2 : f(\mathbf{i}_1, \mathbf{j}_1) \wedge g(\mathbf{i}_2, \mathbf{j}_2) \wedge \mathbf{j}_1 \prec \mathbf{j}_2 \} & \text{if } T_1(\mathbf{j}_1) = T_2(\mathbf{j}_2) \\ \emptyset & \text{otherwise} \end{cases}$$

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
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    }
```

```
O := { S1[i,j] -> [i,j,0,0]; S2[i,j,k] -> [i,j,1,k] };
O << O;
```

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O << O;
```

```
{ S2[i, j, k] -> S2[i', j', k'] : i' >= 1 + i;
  S2[i, j, k] -> S2[i, j', k'] : j' >= 1 + j;
  S2[i, j, k] -> S2[i, j, k'] : k' >= 1 + k;
  S1[i, j] -> S2[i', j', k] : i' >= 1 + i;
  S1[i, j] -> S2[i, j', k] : j' >= 1 + j;
  S1[i, j] -> S2[i, j, k]; S2[i, j, k] -> S1[i', j'] : i' >= 1 + i;
  S2[i, j, k] -> S1[i, j'] : j' >= 1 + j;
  S1[i, j] -> S1[i', j'] : i' >= 1 + i;
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```

Dependence Analysis

Recall: sufficient condition for schedule validity

$$\forall S(\mathbf{i}) \rightarrow T(\mathbf{j}) \in D : O(S(\mathbf{i}), T(\mathbf{j})) < 0$$

Dependence relation D: pairs of statement instances

- accessing the same memory location
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Dependence relation D: pairs of statement instances

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- of which at least one is a write
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Computation:

$$D = \left(\left(W^{-1} \circ R\right) \cup \left(W^{-1} \circ W\right) \cup \left(R^{-1} \circ W\right)\right) \cap \left(O \prec O\right)$$

W: write access relation

R: read access relation

O: original schedule

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
      for (int k = 0; k < K; k++)
S2:       C[i][j] = C[i][j] + A[i][k] * B[k][j];
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W := { S1[i,j] -> C[i,j]; S2[i,j,k] -> C[i,j] };
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I := [M,N,K] -> { S1[i,j] : 0 <= i < M and 0 <= j < N;
      S2[i,j,k] : 0 <= i < M and 0 <= j < N and 0 <= k < K };
O := { S1[i,j] -> [i,j,0,0]; S2[i,j,k] -> [i,j,1,k] };
((R . W^-1) + (W . W^-1) + (W . R^-1)) * (O << O);
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```
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```

Exercise

```
int f1(int m, int n, int A[const static m][n])
{
        int t = 0;
        for (int i = 0; i < m; ++i)
                t += A[i][i];
        return t;
}
void f2(int m, int n, int A[/*.*/m][n][n], int B[/*.*/m][n])
{
        for (int i = 0; i < m; ++i) {
S:              B[i][0] = 0;
                for (int j = 0; j < n; ++j) {
                        if (j == i)
                                continue;
T:                      B[i][j] = f1(j, n, A[i]);
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        }
}
```

Compute dependence relation

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Compute dependence relation {}

Accesses to Structure Fields and Nested Relations

No special treatment is needed for representing accesses to structure fields

$\Rightarrow$ structure field encoded in name of target space of access relations

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```
struct s {
        int a;
        int b[10];
};

void f(struct s s[const static 10][10])
{
        for (int i = 0; i < 10; ++i)
S:              s[i][i].b[9 - i] = 0;
}
```

$\{ \mathrm{S}(i) \rightarrow \mathrm{s\_b}(i, i, 9 - i) \}$

Accesses to Structure Fields and Nested Relations

No special treatment is needed for representing accesses to structure fields

$\Rightarrow$ structure field encoded in name and/or structure of target space of access relations

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{
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S:              s[i][i].b[9 - i] = 0;
}
```

$\{ \mathrm{S}(i) \rightarrow \mathrm{s\_b}(i, i, 9 - i) \}$

In pet, structure encoded in *nested relation*

$\{ \mathrm{S}(i) \rightarrow \mathrm{s\_b}(\mathrm{s}(i, i) \rightarrow \mathrm{b}(9 - i)) \}$

Accesses to Structures

Dependence analysis needs to take into account that access to structure represents access to all fields of structure

```
struct c {
        float re;
        float im;
};

void f(struct c A[const static 10])
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S:      A[0] = A[2];
T:      A[1].re = A[0].im;
}
```

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Write access relation: $\{ \mathtt{S}() \to \mathtt{A}(0); \mathtt{T}() \to \mathtt{A\_re}(\mathtt{A}(1) \to \mathtt{re}()) \}$

Expansion: $\{ \mathtt{A}(a) \to \mathtt{A\_re}(\mathtt{A}(a) \to \mathtt{re}()); \mathtt{A}(a) \to \mathtt{A\_im}(\mathtt{A}(a) \to \mathtt{im}()); \}$

Expanded write access relation:
$\{ \mathtt{S}() \to \mathtt{A\_re}(\mathtt{A}(0) \to \mathtt{re}()); \mathtt{S}() \to \mathtt{A\_im}(\mathtt{A}(0) \to \mathtt{im}());$
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Outline

Schedule Optimization Criteria

Typical optimization criteria

- increase parallelism
- increase locality
- reduce memory requirements

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- increase parallelism
- increase locality
- reduce memory requirements

Parallelism:
Pairs of statement instances $S(\mathbf{i})$ and $T(\mathbf{j})$ may be executed in parallel if they do not depend on each other:

$$\{ S(\mathbf{i}) \rightarrow T(\mathbf{j}); T(\mathbf{j}) \rightarrow S(\mathbf{i}) \} \cap D = \emptyset$$

Local Parallelism

Global parallelism

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Local parallelism

$$\{ S(\mathbf{i}) \to T(\mathbf{j}); T(\mathbf{j}) \to S(\mathbf{i}) \} \cap L = \emptyset$$

- Root of schedule tree: $L = D$
- Child of band node b with partial schedule P:
  $L = L_b \cap \{ S(\mathbf{i}) \to T(\mathbf{j}) : P(S(\mathbf{i})) = P(T(\mathbf{j})) \}$
- Child of sequence node s with instance set F
  $L = L_s \cap \{ S(\mathbf{i}) \to T(\mathbf{j}) : S(\mathbf{i}) \in F \wedge T(\mathbf{j}) \in F \}$

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- Fine grain parallelism (leaf)
  - statement instances $S(\mathbf{i})$ and $T(\mathbf{j})$ for which $O(S(\mathbf{i}), T(\mathbf{j})) = 0$ may be executed in parallel within leaf that contains both
- General case (arbitrary position in schedule tree)
  - schedules defines total order, but,
  - some sequence nodes and/or some band dimensions are explicitly marked parallel

False Dependences

```
for (int i = 0; i < n; ++i) {
S:      t = f1(A[i]);
T:      B[i] = f2(t);
}
```

Dependences

- read after write ("true"): $\{ \mathtt{S}(i) \rightarrow \mathtt{T}(i') : i' \geq i \}$
- write after read ("anti"): $\{ \mathtt{T}(i) \rightarrow \mathtt{S}(i') : i' > i \}$
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False dependences not from dataflow, but from reuse of memory location `t`

Possible solution: expansion/privatization

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for (int i = 0; i < n; ++i) {
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Lexicographic Optimization (Space Local)

- Lexicographic Minimum of Sets (`iscc: lexmin`)

  $S = \{ S(\mathbf{i}) : f(\mathbf{i}) \}$

  $\text{lexmin}\, S = \{ S(\mathbf{i}) : f(\mathbf{i}) \wedge \forall \mathbf{i}' : f(\mathbf{i}') \Rightarrow \mathbf{i} \preccurlyeq \mathbf{i}' \}$

  $I = \{ \mathtt{R}(); \mathtt{S}(i,j) : 0 \leq i < 2 \wedge 0 \leq j < 2; \mathtt{T}(k) : 0 \leq k < 2 \}$

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- Lexicographic Maximum of Relations (iscc: `lexmax`)

  $R = \{ S(\mathbf{i}) \to T(\mathbf{j}) : f(\mathbf{i},\mathbf{j}) \}$
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  $W^{-1} = \{ \mathtt{A}(a) \to \mathtt{S}(i,j) : a = i + j \wedge 0 \le i < 2 \wedge 0 \le j < 2 \}$
  $\operatorname{lexmax}(W^{-1}) = \{ \mathtt{A}(a) \to \mathtt{S}(a,0) : 0 \le a \le 1; \mathtt{A}(2) \to \mathtt{S}(1,1) \}$

  $\Rightarrow$ last statement instance writing array element

Array Dataflow Analysis

Given a read from an array element, what was the last write to the same array element before the read?

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
F:      a[i+j] = f(a[i+j]);
for (i = 0; i < N; ++i)
W:  Write(a[i]);
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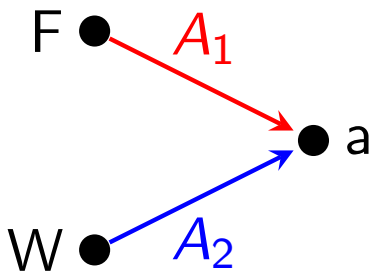

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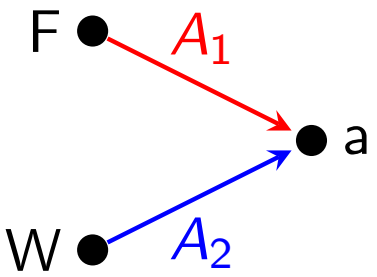

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Last write: $R = \operatorname{lexmax} R' = \{ \mathtt{W}(i) \to \mathtt{F}(i, 0) : 0 \le i < N \}$

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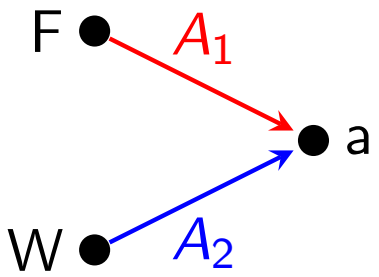

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In general: impose lexicographical order on shared branch of schedule tree

Expansion

Assume:

- instance sets and access relations are static and exact
  $\Rightarrow$ each read has exactly one corresponding write
- single read and write per statement
  $\Rightarrow$ expanded array indexed by statement instance of write

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for (int i = 0; i < n; ++i) {
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$\Rightarrow$ only remaining dependences are dataflow induced

Tagged Access Relations

Expansion in case of multiple reads or writes per statement

$\Rightarrow$ statement instance not enough to identify memory access

$\Rightarrow$ use identifier of array reference instead

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$\Rightarrow$ use identifier of array reference instead

$\Rightarrow$ `pet`: embed array reference in "tagged" instance set

$\Rightarrow$ "ternary" access relations

$$I \to R \to A$$

I: statement instance
R: reference identifier
O: array element

$\Rightarrow$ in practice: nested binary relation

$$(I \to R) \to A$$

For example, $\{ (\mathtt{S}(i) \to \mathtt{R1}()) \to \mathtt{A}(i) \}$

Wrapping, Unwrapping, Domain Map and Range Map

$$R = \{ S(\mathbf{i}) \to T(\mathbf{j}) : f(\mathbf{i},\mathbf{j}) \} \qquad S = \{ (U(\mathbf{i}) \to V(\mathbf{j})) : g(\mathbf{i},\mathbf{j}) \}$$

- Wrap (`iscc: wrap`)

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- Unwrap (iscc: `unwrap`)

  $\mathcal{W}^{-1}S = \{ U(\mathbf{i}) \to V(\mathbf{j}) : g(\mathbf{i},\mathbf{j}) \}$
- Domain map (iscc: `domain_map`)

  $\underrightarrow{\mathrm{dom}}\, R = \{ (S(\mathbf{i}) \to T(\mathbf{j})) \to S(\mathbf{i}) : f(\mathbf{i},\mathbf{j}) \}$

  $I = \{ (\mathtt{S}(i) \to \mathtt{R1}()) : 0 \le i < n \}$

  $\underrightarrow{\mathrm{dom}}(\mathcal{W}^{-1}I) = \{ (\mathtt{S}(i) \to \mathtt{R1}()) \to \mathtt{S}(i) : 0 \le i < n \}$

  $\Rightarrow$ maps tagged instance set to untagged instance set

  $\Rightarrow$ precompose with instance set based relations to obtain tagged instance set based relations

Wrapping, Unwrapping, Domain Map and Range Map

$$R = \{ S(\mathbf{i}) \to T(\mathbf{j}) : f(\mathbf{i},\mathbf{j}) \} \qquad S = \{ (U(\mathbf{i}) \to V(\mathbf{j})) : g(\mathbf{i},\mathbf{j}) \}$$

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- Range map (iscc: `range_map`)

  $\underrightarrow{\mathrm{ran}}\, R = \{ (S(\mathbf{i}) \to T(\mathbf{j})) \to T(\mathbf{j}) : f(\mathbf{i},\mathbf{j}) \}$

Parametric Example: Matrix Multiplication

```
for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
      for (int k = 0; k < K; k++)
S2:       C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
```

- Tagged Access Relations

  $W = \{ (\mathtt{S1}(i,j) \to \mathtt{R0}()) \to \mathtt{C}(i,j); (\mathtt{S2}(i,j,k) \to \mathtt{R1}()) \to \mathtt{C}(i,j) \}$

  $R = \{ (\mathtt{S2}(i,j,k) \to \mathtt{R2}()) \to \mathtt{C}(i,j); (\mathtt{S2}(i,j,k) \to \mathtt{R3}()) \to \mathtt{A}(i,k);$
  $(\mathtt{S2}(i,j,k) \to \mathtt{R4}()) \to \mathtt{B}(k,j) \}$

- Tagged Schedule

  $\{ (\mathtt{S1}(i,j) \to \mathtt{R0}()) \to (i,j,0,0); (\mathtt{S2}(i,j,k) \to \mathtt{R1}()) \to (i,j,1,k)$
  $(\mathtt{S2}(i,j,k) \to \mathtt{R2}()) \to (i,j,1,k); (\mathtt{S2}(i,j,k) \to \mathtt{R3}()) \to (i,j,1,k);$
  $(\mathtt{S2}(i,j,k) \to \mathtt{R4}()) \to (i,j,1,k) \}$

- Tagged Dataflow

  $\{ (\mathtt{S1}(i,j) \to \mathtt{R0}()) \to (\mathtt{S2}(i,j,0) \to \mathtt{R2}())$
  $(\mathtt{S2}(i,j,k) \to \mathtt{R1}()) \to (\mathtt{S2}(i,j,k+1) \to \mathtt{R2}()) \}$

Maximal Static Expansion

[3]

```
for (int i = 0; i < n; ++i) {
S1:       t = f1(i);
S2:       A[i] = t;
S3:       t = f2(i);
S4:       if (f3(i))
S5:             t = f4(i);
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Dataflow cannot be determined independently of run-time information

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Dataflow cannot be determined independently of run-time information

$\Rightarrow$ approximate dataflow

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$\Rightarrow$ corresponding equivalence classes should not be expanded apart

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[3]

```
for (int i = 0; i < n; ++i) {
S1:       t = f1(i);                    t1[i] = f1(i);
S2:       A[i] = t;                     A[i] = t1[i];
S3:       t = f2(i);                    t2[i] = f2(i);
S4:       if (f3(i))                    if (f3(i))
S5:           t = f4(i);                    t2[i] = f4(i);
S6:       B[i] = t;                     B[i] = t2[i];
}
```

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Approximate Dataflow Analysis

How to compute dataflow in presence of data dependent control?

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- Direct computation
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May Writes

Keep track of whether write is possible or definite

- Must-writes
  Array elements that are definitely accessed by statement instance
- May-writes
  Array elements that are possibly accessed by statement instance

Must-write access relation is subset of may-write access relation

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  Array elements that are definitely accessed by statement instance
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  Array elements that are possibly accessed by statement instance
  - statement instance not necessarily executed

```
for (i = 0; i < n; ++i)
    if (A[i] > 0)
S:      B[i] = A[i];
```

May-write: $\{ \mathtt{S}(i) \rightarrow \mathtt{B}(i) \}$

Must-write access relation is subset of may-write access relation

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for (i = 0; i < n; ++i)
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S:      B[i] = A[i];
```

    May-write: $\{ \mathtt{S}(i) \rightarrow \mathtt{B}(i) \}$
  - array element not necessarily accessed

```
int A[N];
/* ... */
T:  A[B[0]] = 5;
```

    May-write: $\{ \mathtt{T}() \rightarrow \mathtt{A}(a) : 0 \leq a < N \}$

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- Read after write dependences
  - write and read access same memory location
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    $\Rightarrow$ intermediate write kills dependence

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Approximate Dataflow — Direct Computation

- Read after write dependences
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  $\Rightarrow$ Approximate dataflow analysis with no must-writes
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Performing Dataflow Analysis

[9, 17]

Two possibilities

1. dataflow analysis on polyhedral model
   - first extract instance set, access relations and schedule
   - then perform dataflow analysis

   E.g., `isl`
2. dataflow analysis on AST before/during model extraction
   Proposed by, e.g., Maslov (1994)

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Dataflow in Parallel Programs

1. use refined execution order during dataflow analysis
2. remove spurious dependences after dataflow analysis

   Note: dataflow from must-writes cannot be removed without caution

   $\Rightarrow$ must-write may have killed other dependences
   $\Rightarrow$ other dependences may have to be added back

Explicit Kills

```
     int A[N];

S:   A[0] = 1;

     for (int i = 0; i < N; ++i)
T:       A[perm[i]] = f(i);
U:   f(A[0]);
```

- Assume `perm` represents a permutation

  $\Rightarrow$ there can be no dataflow from `S` to `U`

Explicit Kills

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- Assume `perm` represents a permutation
  - $\Rightarrow$ there can be no dataflow from `S` to `U`
- Compiler does not know all elements are written by `T`
  - $\Rightarrow$ may find dataflow from `S` to `U`
  - $\Rightarrow$ user can insert explicit kill
  - $\Rightarrow$ explicit kill used to kill dependences, just like must-write

Explicit Kills

```
     int A[N];

S:   A[0] = 1;
     __pencil_kill(A);
     for (int i = 0; i < N; ++i)
T:       A[perm[i]] = f(i);
U:   f(A[0]);
```

- Assume `perm` represents a permutation
  - $\Rightarrow$ there can be no dataflow from `S` to `U`
- Compiler does not know all elements are written by `T`
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Local Variables and Kills

```
for (int i = 0; i < N; ++i) {
    int t;
    /* ... */
}
```

⇒ there can be no dataflow on `t` across different iterations

⇒ pet automatically inserts kills

- before declaration and
- at end of block containing declaration

Killing False Dependences

Dataflow derived from read after write dependences through killing.

Should we do the same for false dependences?

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Optimizing locality

- typical criterion: minimize maximal dependence distance

  dependence distances: $\{ P(S(\mathbf{i})) - P(T(\mathbf{j})) : S(\mathbf{i}) \to T(\mathbf{j}) \in D \}$

  D: (local) dependence relation, P: partial schedule

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  dependence distances: $\{ P(S(\mathbf{i})) - P(T(\mathbf{j})) : S(\mathbf{i}) \to T(\mathbf{j}) \in D \}$

  D: (local) dependence relation, P: partial schedule
- may also be useful for false dependences if no expansion is performed

  $\Rightarrow$ locality of reused memory location
- killing false dependences avoids critical path determined by transitively covered dependences

  $\Rightarrow$ allow must-writes to kill dependences, but not explicit kills

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Run-time Dependent Dataflow Analysis

[4, 25]

Approaches

- “fuzzy array dataflow analysis”
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- Run-time dependent dataflow

  $\{ \mathtt{S1}(i) \rightarrow \mathtt{S2}(i); \mathtt{S3}(i) \rightarrow \mathtt{S6}(i) : \beta^{\mathtt{S5}}_{\mathtt{S6}} = 0; \mathtt{S5}(i) \rightarrow \mathtt{S6}(i) : \beta^{\mathtt{S5}}_{\mathtt{S6}} = 1 \}$

  β^P_C: any potential source instance P is executed for sink C

  $\boldsymbol{\lambda}^P_C$: last potential source instance P executed for sink C

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  β^P_C: any potential source instance P is executed for sink C

  $\boldsymbol{\lambda}^P_C$: last potential source instance P executed for sink C
- Approximate dataflow (project out β and $\boldsymbol{\lambda}$)

  $\{\, \mathtt{S1}(i) \rightarrow \mathtt{S2}(i); \mathtt{S3}(i) \rightarrow \mathtt{S6}(i); \mathtt{S5}(i) \rightarrow \mathtt{S6}(i) \,\}$

Representing Dynamic Conditions

```
N1:  n = f();
     for (int k = 0; k < 100; ++k) {
M:       m = g();
         for (int i = 0; i < m; ++i)
             for (int j = 0; j < n; ++j)
A:               a[j][i] = g();
N2:      n = f();
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```

What is instance set (restricted to A statement)?

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$\{\mathtt{A}(k,i,j) : 0 \leq k < 100 \wedge 0 \leq i < \mathtt{valueOf\_m}(k) \wedge 0 \leq j < \mathtt{valueOf\_n}(k)\}$?

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Alternative: use overapproximation of instance set and keep track of which elements are executed

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     }
```

- Instance set: $\{ \mathtt{A}(k,i,j) : 0 \leq k < 100 \wedge 0 \leq i \wedge 0 \leq j \}$
- Filter:
  - Filter access relations: reader $\rightarrow$ (writer $\rightarrow$ array element)
    - $F_1^{\mathtt{A}} = \{ \mathtt{A}(k,i,j) \rightarrow (\mathtt{M}(k) \rightarrow \mathtt{m}()) \}$
    - $F_2^{\mathtt{A}} = \{ \mathtt{A}(0,i,j) \rightarrow (\mathtt{N1}() \rightarrow \mathtt{n}()); \mathtt{A}(k,i,j) \rightarrow (\mathtt{N2}(k-1) \rightarrow \mathtt{n}()) : k \geq 1 \}$
  - Filter value relation:
    $V^{\mathtt{A}} = \{ \mathtt{A}(k,i,j) \rightarrow (m,n) : 0 \leq k \leq 99 \wedge 0 \leq i < m \wedge 0 \leq j < n \}$

Statement instance is executed iff values written by corresponding write accesses (through filter access relations) satisfy filter value relation

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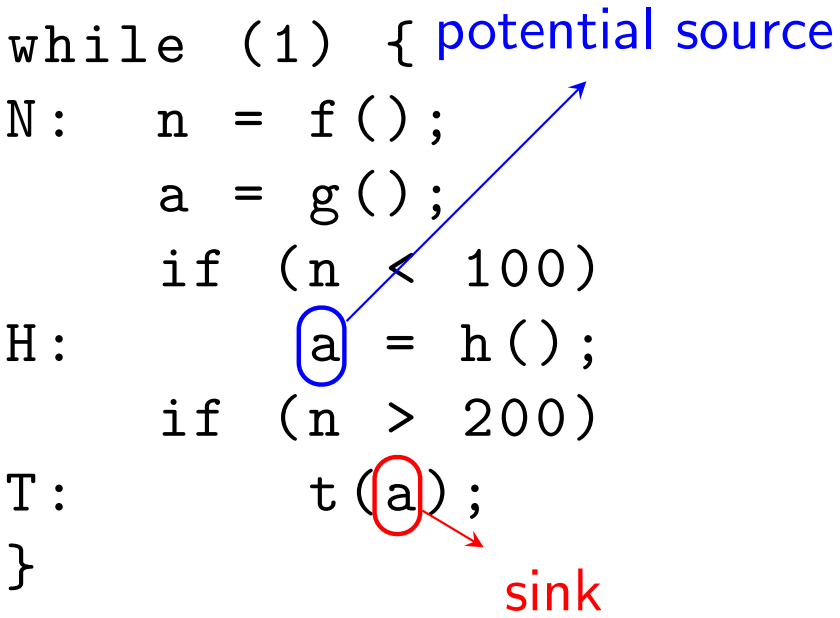
Parametric Array Dataflow Analysis



Is there any dataflow between potential source and sink at inner level?

Parametric Array Dataflow Analysis

```
while (1) {
N:   n = f();
     a = g();
     if (n < 100)
H:       a = h();
     if (n > 200)
T:       t(a);
}
```

potential source

sink

$$I = \{\mathtt{H}(i) : i \geq 0; \mathtt{T}(i) : i \geq 0\}$$
$$F^{\mathtt{H}} = \{\mathtt{H}(i) \rightarrow (\mathtt{N}(i) \rightarrow \mathtt{n}())\}$$
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$$F^{\mathtt{T}} = \{\mathtt{T}(i) \rightarrow (\mathtt{N}(i) \rightarrow \mathtt{n}())\}$$
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Is there any dataflow between potential source and sink at inner level?

- $M = \{ \mathtt{T}(i) \rightarrow \mathtt{H}(i) \}$

Parametric Array Dataflow Analysis

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while (1) {
N:   n = f();
     a = g();
     if (n < 100)
H:       a = h();
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T:       t(a);
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```

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- $\left(V^{\mathtt{T}} \circ M^{-1}\right) \cap V^{\mathtt{H}} = \emptyset$
  - $\Rightarrow$ there can be no dataflow at inner level

Polyhedral Process Networks

[19]

- Main purpose: extract task level parallelism from dataflow graph

$$\begin{aligned} \text{statement} &\rightarrow \text{process} \\ \text{flow dependence} &\rightarrow \text{communication channel} \end{aligned}$$

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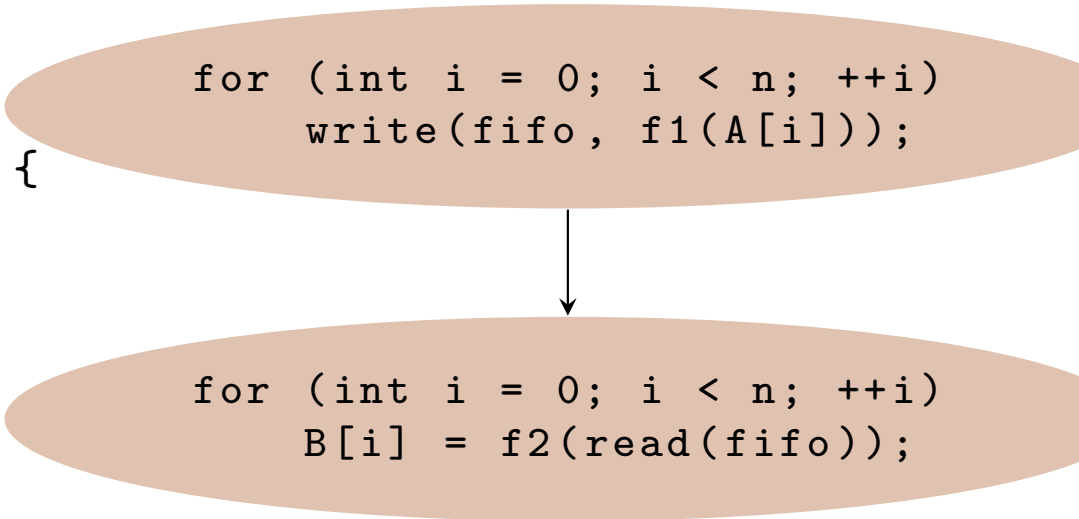

Process Networks with Dynamic Control

```
for (int i = 0; i < n; ++i) {
S1:       t = f1(i);
S2:       A[i] = t;
S3:       t = f2(i);
S4:       if (f3(i))
S5:                 t = f4(i);
S6:       B[i] = t;
}
```

Run-time dependent dataflow:

$$\{\, \mathrm{S1}(i) \to \mathrm{S2}(i); \mathrm{S3}(i) \to \mathrm{S6}(i) : \beta^{\mathrm{S5}}_{\mathrm{S6}} = 0;$$
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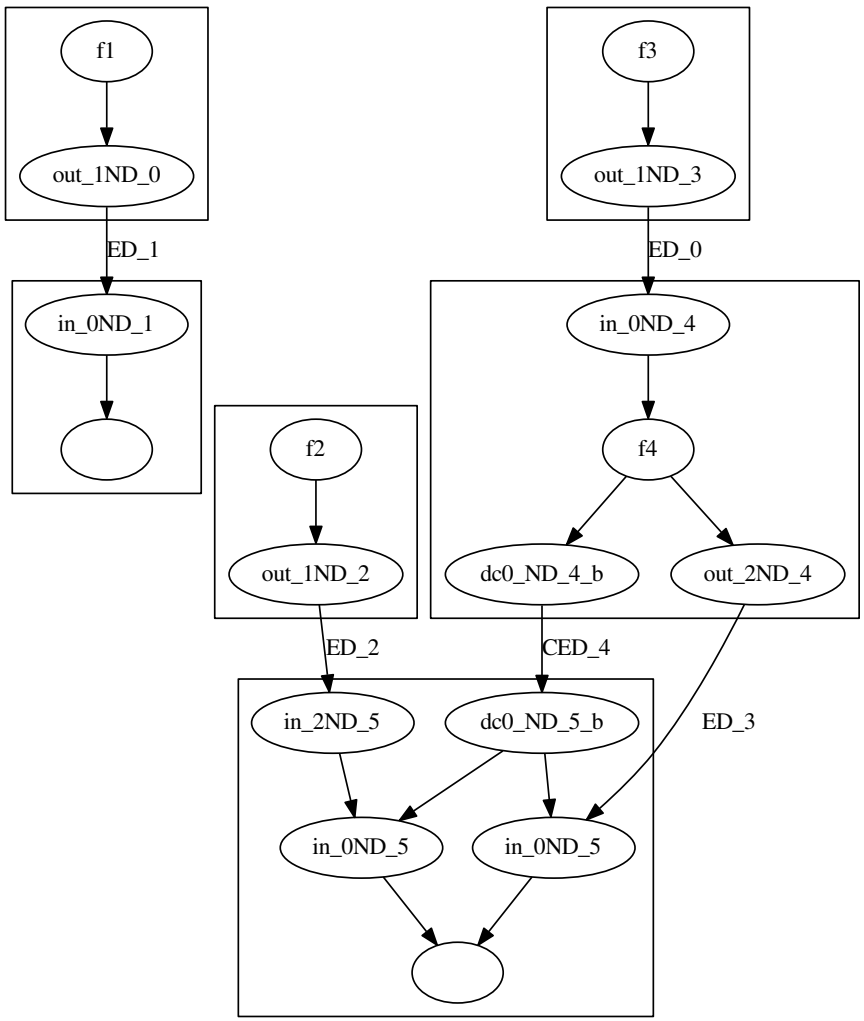

Reductions

```
A:   s = 0;
     for (int i = 0; i < n; ++i)
B:       s += A[i];
C:   B[0] = s;
```

Dataflow:

$\{ \mathtt{A}() \to \mathtt{B}(0); \mathtt{B}(i) \to \mathtt{B}(i+1) : 0 \leq i < n-1; \mathtt{B}(n-1) \to \mathtt{C}() \}$

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Outline

Aliasing

Some possible ways of handling aliasing:

- use an input language that does not permit aliasing
- pretend the problem does not exist
- require user to ensure absence of aliasing
  $\Rightarrow$ e.g., use `restrict` keyword
- handle as may-write
  $\Rightarrow$ may lead to too many dependences
- check aliasing at run-time
  $\Rightarrow$ use original code in case of aliasing

Outline

Cardinality

- Cardinality of a set
  - $\Rightarrow$ number of elements in the set
  - $\Rightarrow$ may depend on symbolic constants

$S = \{ S(\mathbf{i}) : f(\mathbf{i}) \}$

$\operatorname{card} S = \{ n : n = \#\mathbf{i} : f(\mathbf{i}) \}$ $\qquad \operatorname{card}\left(\bigcup_i S_i\right) := \sum_i \operatorname{card} S_i$

$\operatorname{card} \{ \mathtt{A}(i) : 0 \leq i \leq n; \mathtt{B}() \} = n + 2$

- Cardinality of a binary relation
  - $\Rightarrow$ for each domain element, number of corresponding images

$R = \{ S(\mathbf{i}) \to T(\mathbf{j}) : f(\mathbf{i}, \mathbf{j}) \}$

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$\operatorname{card} R = \{ \mathtt{A}(i) \to 1 : 0 \leq i \leq n; \mathtt{B}() \to n + 1 \}$

$\Rightarrow$ not a Presburger formula

Cardinality Examples

```
for (i = 0; i < N; ++i)
    for (j = 0; j < N - i; ++j)
        a[i+j] = f(a[i+j]);
```

- How many times is the statement executed?

  $\mathrm{card}\{(i,j) : 0 \leq i < N \wedge 0 \leq j < N - i\}$

  $\Rightarrow \{\frac{N+N^2}{2} : N \geq 1\}$

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Cardinality Examples (2)

How many times is S1 executed ?

```
for (i = max(0,N-M); i <= N-M+3; i++)
    for (j = 0; j <= N-2*i; j++)
        S1;
```

$$\mathrm{card}\{(i,j) : 0, N-M \leq i \leq N-M+3 \wedge 0 \leq j \leq N-2i\}$$

$$\begin{cases} -4N+8M-8 & \text{if } M \leq N \leq 2M-6 \\ MN-2N-M^2+6M-8 & \text{if } N \leq M \leq N+3 \wedge N \leq 2M \\ \frac{N^2}{4}+\frac{3}{4}N+\frac{1}{2}\left\lfloor\frac{N}{2}\right\rfloor+1 & \text{if } 0 \leq N \leq M \wedge 2M \leq N+6 \\ \frac{N^2}{4}-MN-\frac{5}{4}N+M^2+2M+\frac{1}{2}\left\lfloor\frac{N}{2}\right\rfloor+1 & \text{if } M \leq N \leq 2M \leq N+6 \end{cases}$$

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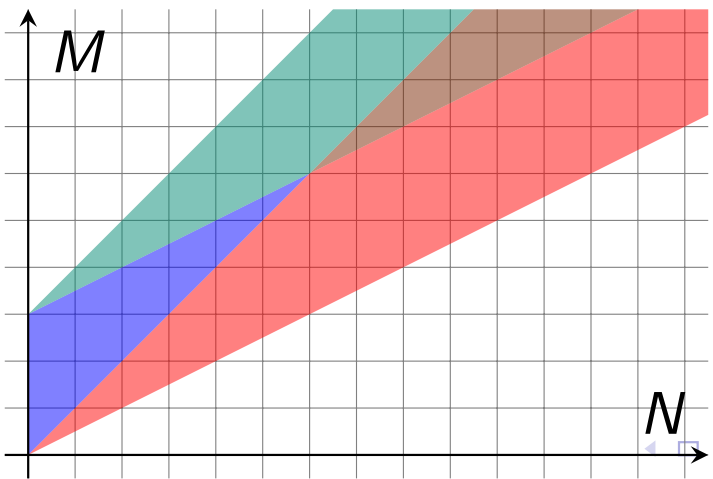

Cardinality Representation

- Integer quasi affine expression $\lfloor x/2 \rfloor + 3N$

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Note: in practice, cardinality result does not contain nested integer divisions

Basic Operations on Piecewise Expressions

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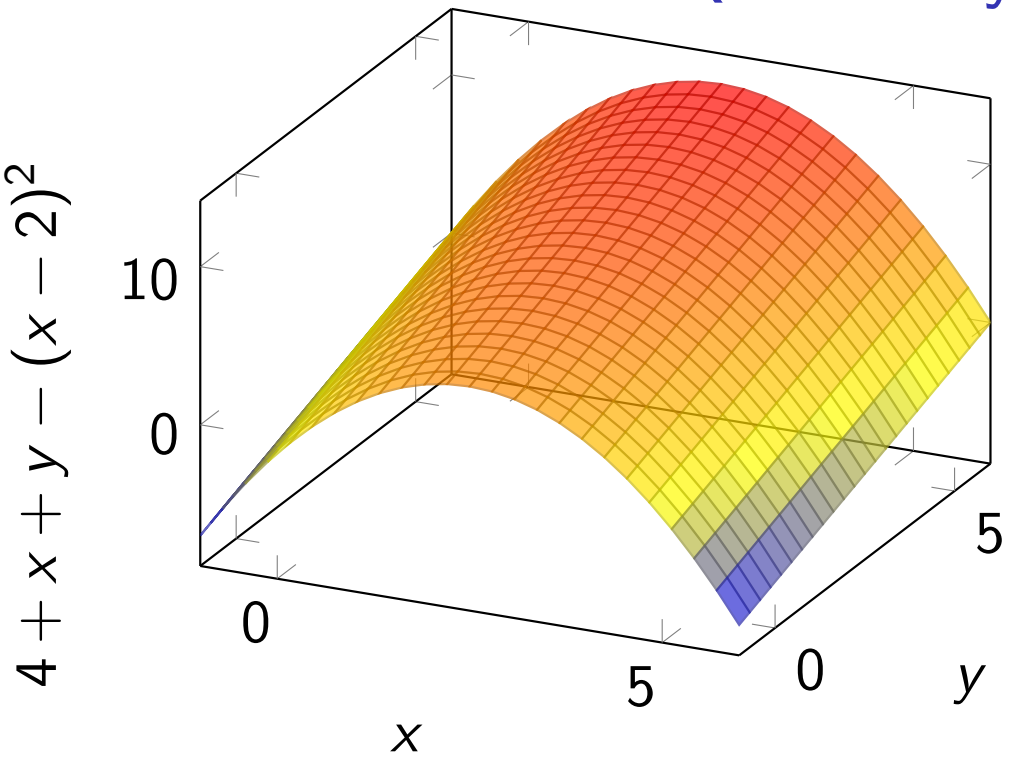

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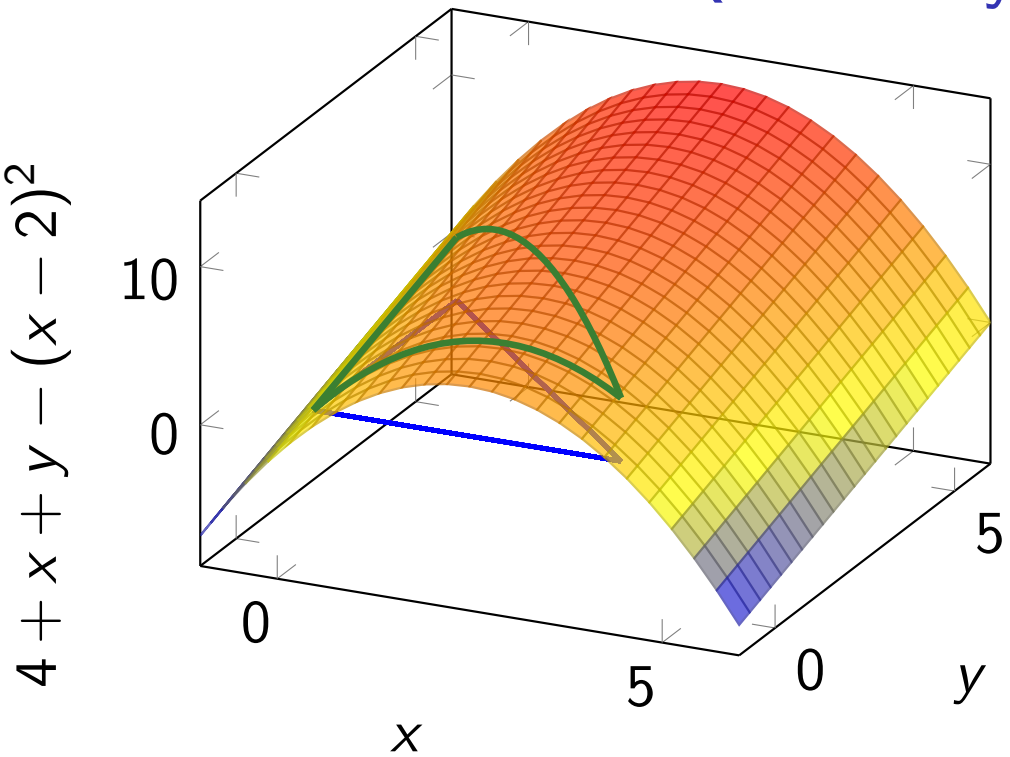

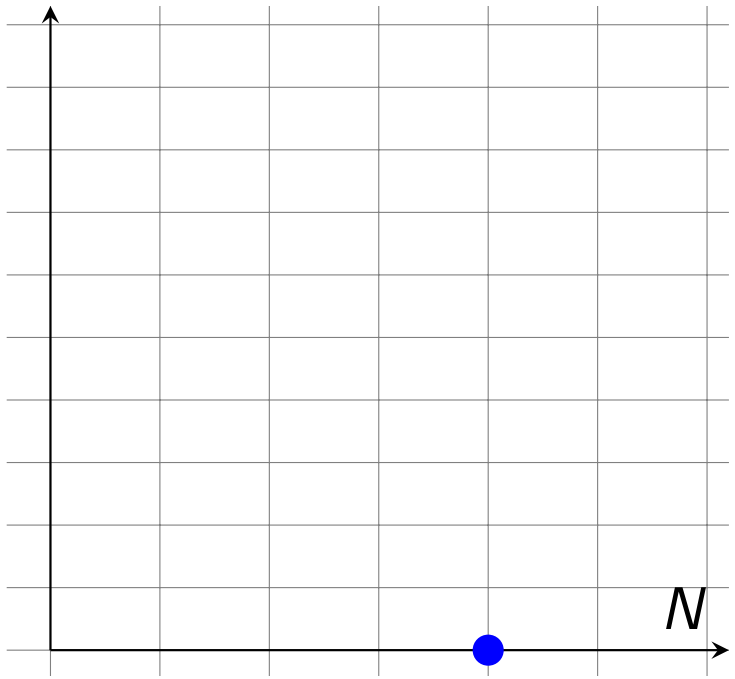

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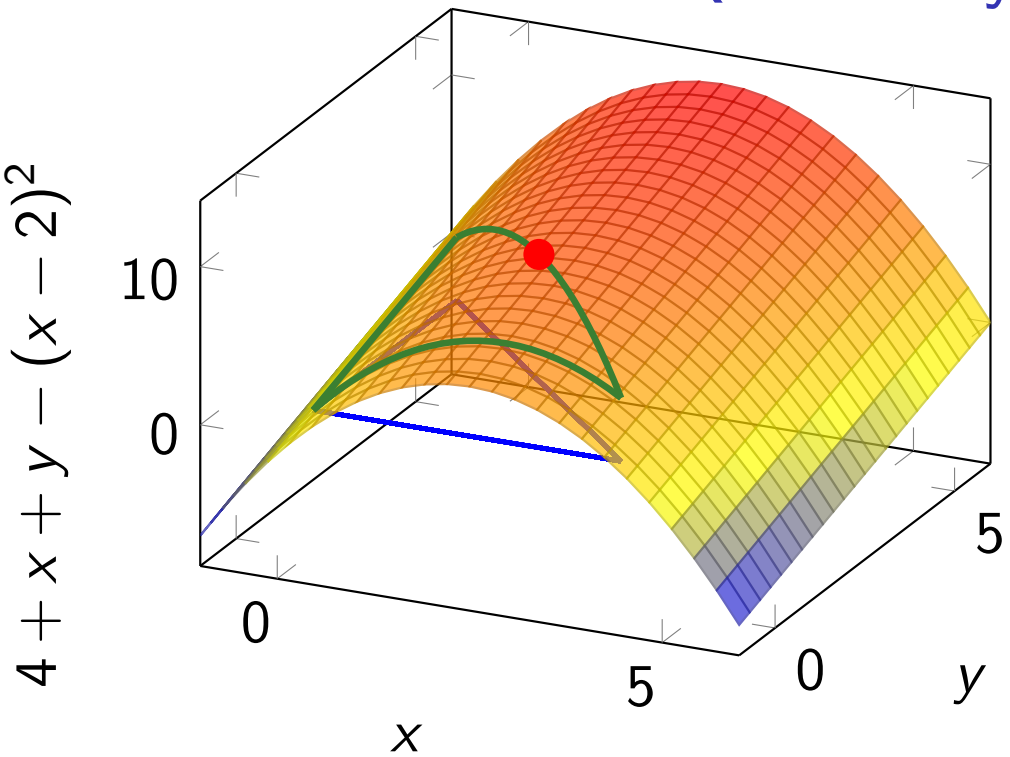

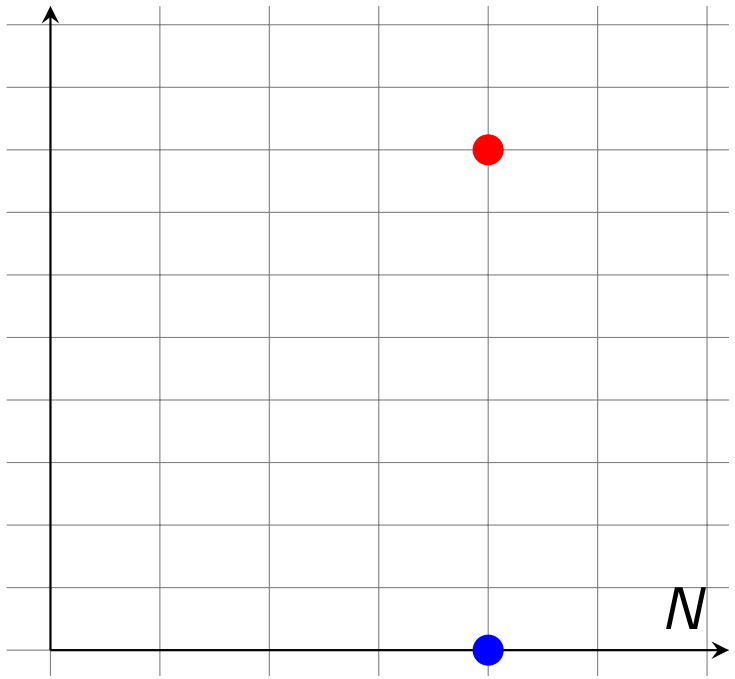

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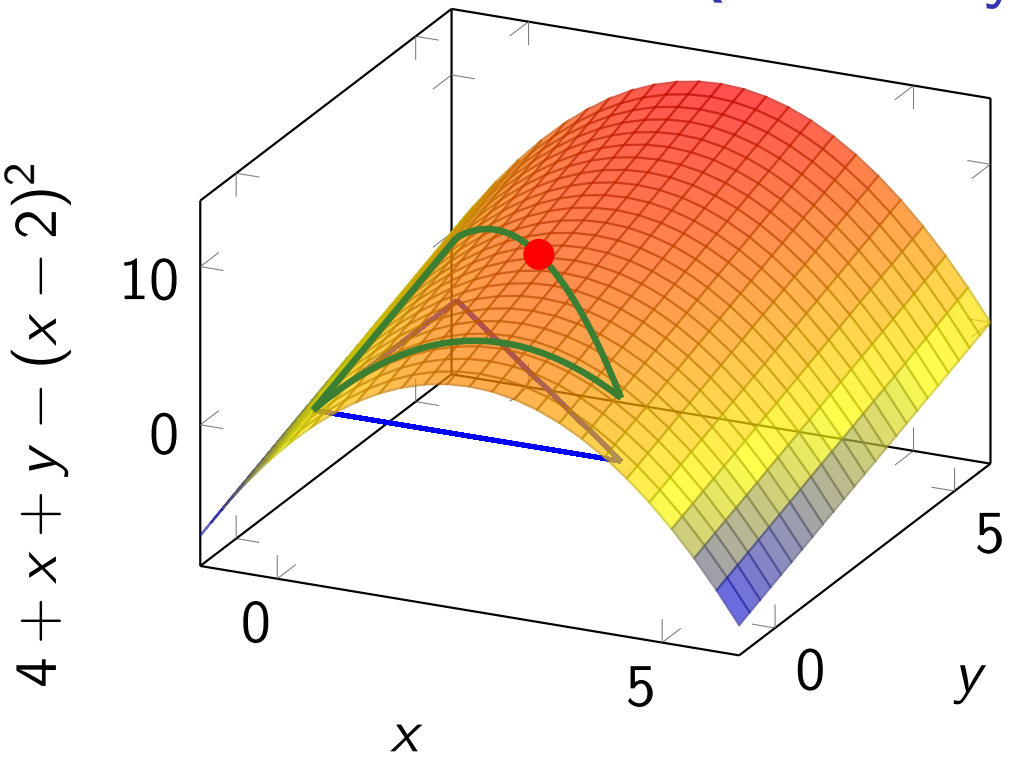

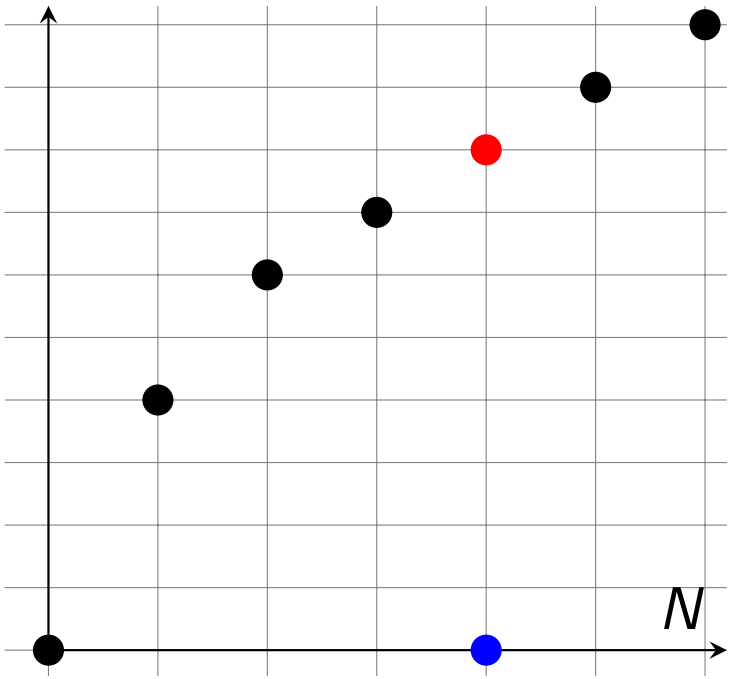

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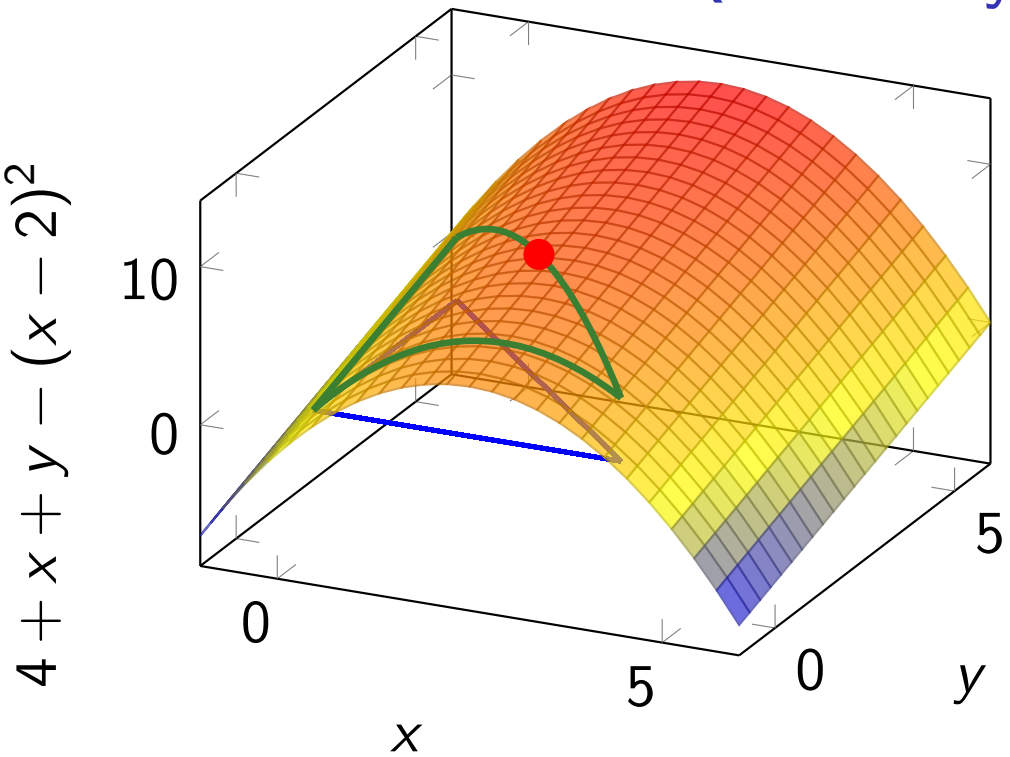

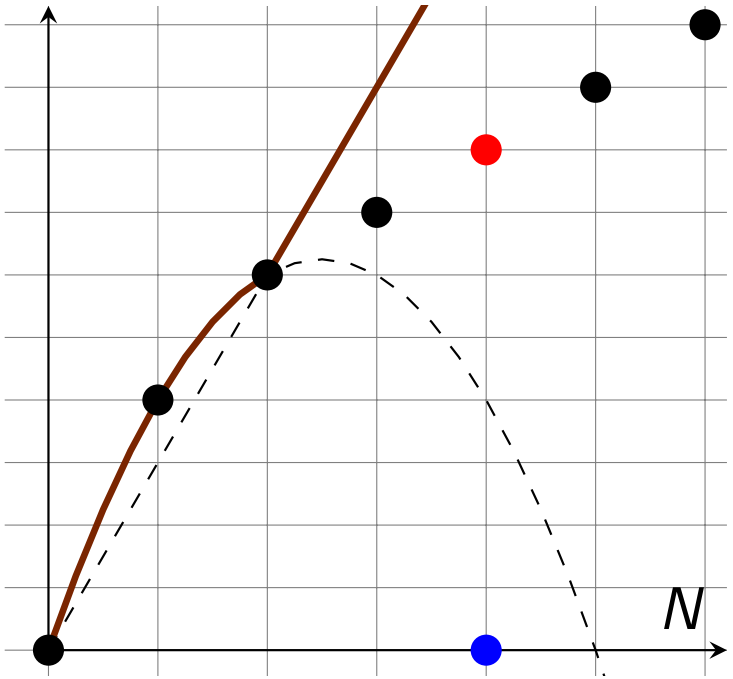

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Question

Can exact maximum be computed in general?

Upper bound $u(N) \geq m(N)$ can be computed

Bounds on Piecewise Quasi Polynomials — Example

```
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j) {
        p = malloc(i * j + i - N + 1);
        /* ... */
        free(p);
    }
```

How much memory is needed?

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Result:

$$\begin{cases} \max(1 - 2N + N^2) & \text{if } N \ge 1 \end{cases}$$

(exact maximum)

Maximal Number of Live Memory elements

- Assume each statement instance writes to at most one array element
  $\Rightarrow$ Each live element can be identified by write instance
- Compute dataflow relation D
- For each write instance compute last read
  $L = O^{-1} \circ \operatorname{lexmax}(O \circ D)$
- For each statement instance **i**, count write instances that precede **i** such that corresponding last read follows **i**
  $\Rightarrow$ Number of live elements at **i**
  $N = \operatorname{card}\left(((O \succ O) \cap_{ran} (\operatorname{dom} L)) \cap \left(L^{-1} \circ (O \preccurlyeq O)\right)\right)$
- Compute upper bound
  $U = \operatorname{ub} N$

Maximal Number of Live Memory elements — Example

```
for (i = 0; i < N; ++i)
S1:     t[i] = f(a[i]);
for (i = 0; i < N; ++i)
S2:     b[i] = g(t[N-i-1]);
```

$$I = \{ \mathrm{S1}(i) : 0 \leq i < N; \mathrm{S2}(i) : 0 \leq i < N \}$$

$$O = \{ \mathrm{S1}(i) \rightarrow (0, i); \mathrm{S2}(i) \rightarrow (1, i) \}$$

$$D = \{ \mathrm{S1}(i) \rightarrow \mathrm{S2}(N - 1 - i) : 0 \leq i < N \}$$

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$$\begin{aligned} L &= O^{-1} \circ \mathrm{lexmax}(O \circ D) \\ &= \{\, \mathrm{S1}(i) \to \mathrm{S2}(N - 1 - i) : 0 \le i < N \,\} \end{aligned}$$

$$\begin{aligned} N &= \mathrm{card}\left(((O \succ O) \cap_{\mathsf{ran}} (\mathrm{dom}\, L)) \cap \left(L^{-1} \circ (O \preccurlyeq O)\right)\right) \\ &= \{\, \mathrm{S1}(i) \to i : 1 \le i < N; \mathrm{S2}(i) \to N - i : 0 \le i < N \,\} \end{aligned}$$

$$\begin{aligned} U &= \mathrm{ub}\, N \\ &= \{\, \mathsf{max}(N) : N \ge 1 \,\} \end{aligned}$$

Weighted Counting

$$G = F \circ R$$

with F a piecewise quasi polynomial and R a Presburger relation is a piecewise quasi polynomial G such that

$$G(\mathbf{i}) = \sum_{\mathbf{j}:R(\mathbf{i},\mathbf{j})} F(\mathbf{j})$$

Weighted Counting

$$G = F \circ R = \left\{ (x, y) \to \frac{x^2 + y^2}{4} : 1 \leq x, y \leq 2 \right\} \circ \{ (x) \to (x, y) \}$$

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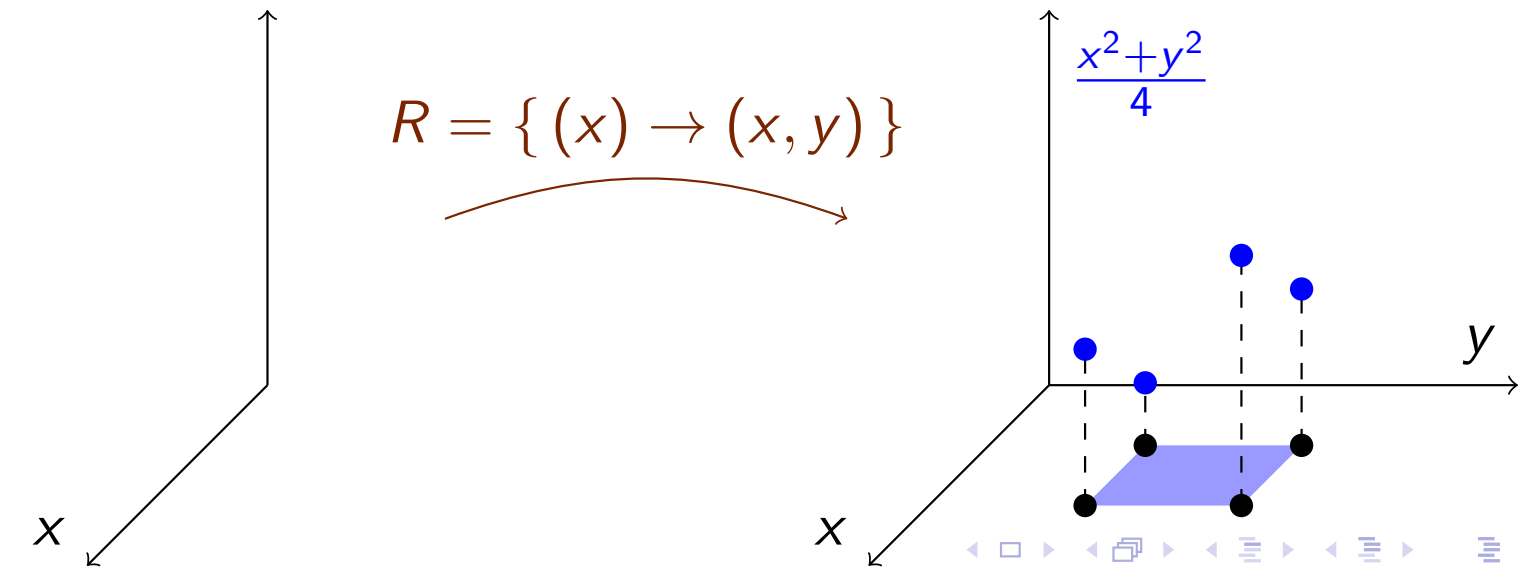

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$$= \left\{ (x) \to \frac{5+2x^2}{2} : 1 \le x \le 2 \right\}$$

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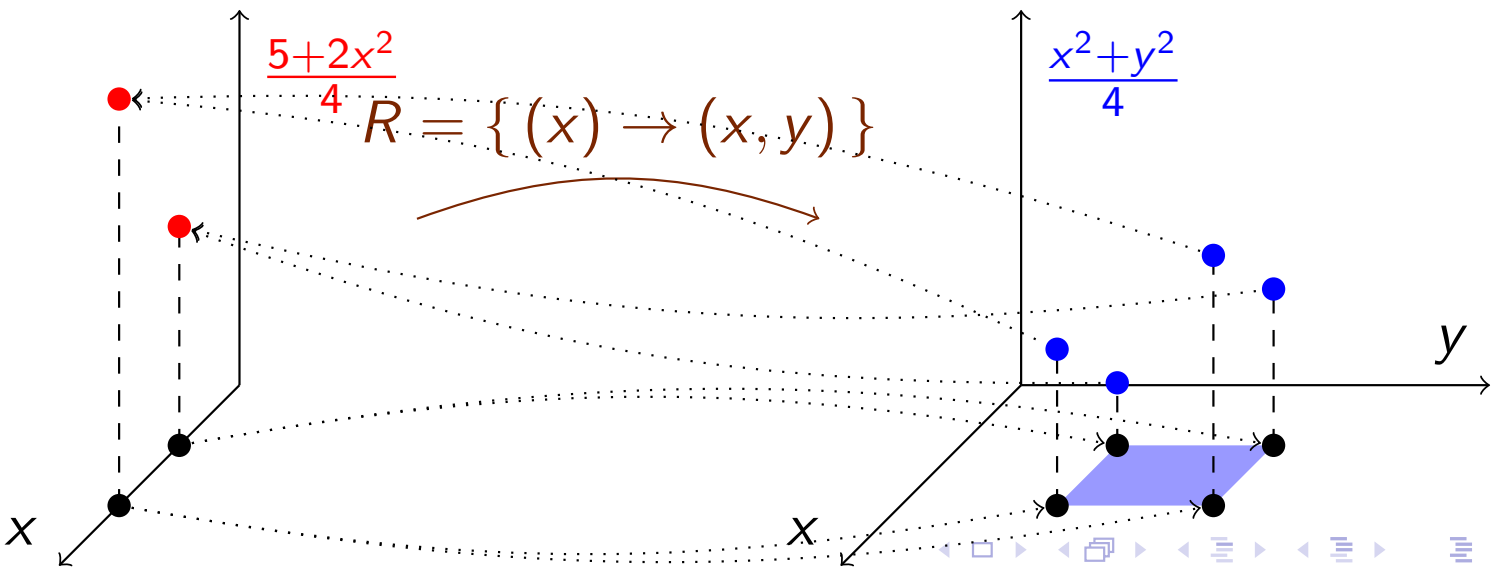

Compositions with Piecewise (Folds of) Quasi polynomials

$$F \circ R$$

- R: $D_1 \to D_2$ is a Presburger relation
- F: $D_2 \to \mathbb{Q}$ may be
  - piecewise quasi polynomial
    (result of counting problem)

    $\Rightarrow$ take sum over $(\operatorname{ran} R) \cap (\operatorname{dom} F)$
  - piecewise fold of quasi polynomials
    (result of upper bound computation)

    $\Rightarrow$ compute bound over $(\operatorname{ran} R) \cap (\operatorname{dom} F)$
- $(F \circ R)$: $D_1 \to \mathbb{Q}$ of same type as F

if R is single-valued, then sum/bound is computed over a single point

Compositions with Piecewise (Folds of) Quasi polynomials

$$F \circ R \qquad \text{or} \qquad F(S)$$

- R: $D_1 \to D_2$ is a Presburger relation
- $S \subseteq D_2$ is a Presburger set
- F: $D_2 \to \mathbb{Q}$ may be
  - piecewise quasi polynomial
    (result of counting problem)

    $\Rightarrow$ take sum over $(\operatorname{ran} R) \cap (\operatorname{dom} F)$ or $S \cap (\operatorname{dom} F)$
  - piecewise fold of quasi polynomials
    (result of upper bound computation)

    $\Rightarrow$ compute bound over $(\operatorname{ran} R) \cap (\operatorname{dom} F)$ or $S \cap (\operatorname{dom} F)$
- $(F \circ R)$: $D_1 \to \mathbb{Q}$ of same type as F
- $F(S)$: $\mathbb{Q}$ of same type as F

if R is single-valued, then sum/bound is computed over a single point

Example: Total Memory Allocation

```
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        p[i][j] = malloc(i * j + i - N + 1);
/* ... */
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        free(p[i][j]);
```

How much memory allocated in total?

Example: Total Memory Allocation

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for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        p[i][j] = malloc(i * j + i - N + 1);
/* ... */
for (i = 0; i < N; ++i)
    for (j = i; j < N; ++j)
        free(p[i][j]);
```

How much memory allocated in total?

$$F = \{ (i,j) \to ij + i - N + 1 \}$$

$$I = \{ (i,j) : 0 \leq i < N \wedge i \leq j < N \}$$

$$F(I) = \left\{ \frac{5}{12}N - \frac{1}{8}N^2 - \frac{5}{12}N^3 + \frac{1}{8}N^4 : N \geq 1 \right\}$$

[7]

Dynamic Memory Requirement Estimation

How much memory is needed to execute the following program?

```
void m0(int m) {
  for (c = 0; c < m; c++) {
    m1(c);                    /*S1*/
    B[] m2Arr = m2(2*m-c);/*S2*/
  }
}
void m1(int k) {
  for (i = 1; i <= k; i++) {
    A a = new A();            /*S3*/
    B[] dummyArr = m2(i); /*S4*/
  }
}
B[] m2(int n) {
  B[] arrB = new B[n];      /*S5*/
  for (j = 1; j <= n; j++)
    B b = new B();            /*S6*/
  return arrB;
}
```

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$$\mathtt{m1}(k) \rightarrow \mathtt{S4}(i) : 1 \leq i \leq k;$$
$$\mathtt{m2}(n) \rightarrow \mathtt{S5}();$$
$$\mathtt{m2}(n) \rightarrow \mathtt{S6}(j) : 1 \leq j \leq n \}$$

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Dynamic Memory Requirement Estimation

How much (scoped) memory is needed?

$\Rightarrow$ compute for each method

$\mathrm{ret}_{\mathtt{m}}$ size of memory returned by `m`

$\mathrm{cap}_{\mathtt{m}}$ size of memory "captured" (not returned) by `m`

$\mathrm{memRq}_{\mathtt{m}}$ total memory requirements of `m`

$$\mathrm{ret}_{\mathtt{m}} + \mathrm{cap}_{\mathtt{m}} = \sum_{\mathtt{p} \text{ called by } \mathtt{m}} \mathrm{ret}_{\mathtt{p}}$$

$$\mathrm{memRq}_{\mathtt{m}} = \mathrm{cap}_{\mathtt{m}} + \max_{\mathtt{p} \text{ called by } \mathtt{m}} \mathrm{memRq}_{\mathtt{p}}$$

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$\Rightarrow$ summarize over iteration domain, i.e., compose with $M = (\underrightarrow{\mathrm{dom}}\, I)^{-1}$

$$\begin{aligned} M = \{ & \mathtt{m0}(m) \to (\mathtt{m0}(m) \to \mathtt{S1}(c)) : 0 \le c < m; \\ & \mathtt{m0}(m) \to (\mathtt{m0}(m) \to \mathtt{S2}(c)) : 0 \le c < m; \\ & \mathtt{m1}(k) \to (\mathtt{m1}(k) \to \mathtt{S3}(i)) : 1 \le i \le k; \\ & \mathtt{m1}(k) \to (\mathtt{m1}(k) \to \mathtt{S4}(i)) : 1 \le i \le k; \\ & \mathtt{m2}(n) \to (\mathtt{m2}(n) \to \mathtt{S5}()); \mathtt{m2}(n) \to (\mathtt{m2}(n) \to \mathtt{S6}(j)) : 1 \le j \le n \} \end{aligned}$$

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Dynamic Memory Requirement Estimation

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Dynamic Memory Requirement Estimation

$$\mathrm{ret_m} + \mathrm{cap_m} = \sum_{\mathtt{p} \text{ called by } \mathtt{m}} \mathrm{ret_p}$$

$$\mathrm{memRq_m} = \mathrm{cap_m} + \max_{\mathtt{p} \text{ called by } \mathtt{m}} \mathrm{memRq_p}$$

```
B[] m2(int n) {
 B[] arrB = new B[n];        /*S5*/
 for (j = 1; j <= n; j++)
  B b = new B();             /*S6*/
 return arrB;
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```

$$\mathrm{ret}_{\mathtt{m2}} = \{\, (\mathtt{m2}(n) \to \mathtt{S5}()) \to n : n \geq 0 \,\} \circ M$$

$$\mathrm{cap}_{\mathtt{m2}} = \{\, (\mathtt{m2}(n) \to \mathtt{S6}(j)) \to 1 \,\} \circ M$$

$$\mathrm{memRq}_{\mathtt{m2}} = \mathrm{cap}_{\mathtt{m2}} + \{\, \mathtt{m2}(n) \to \max(0) \,\}$$

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Dynamic Memory Requirement Estimation

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Dynamic Memory Requirement Estimation

```
void m1(int k) {
    for (i = 1; i <= k; i++) {
        A a = new A();                    /* S3 */
        B[] dummyArr = m2(i);             /* S4 */
    }
}
```

$$\mathrm{cap}_{\mathtt{m1}}(k) = \sum_{1 \le i \le k} (1 + \mathrm{ret}_{\mathtt{m2}}(i))$$

$\mathrm{ret}_{\mathtt{m2}}$ is a function of the arguments of `m2`

We want to use it as a function of the arguments and local variables of `m1`

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$\Rightarrow$ define parameter binding

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Dynamic Memory Requirement Estimation

How much memory is needed to execute the following program?

```
void m0(int m) {
  for (c = 0; c < m; c++) {
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  for (j = 1; j <= n; j++)
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  return arrB;
}
```

$$
\begin{aligned}
I = \{ & \mathtt{m0}(m) \rightarrow \mathtt{S1}(c) : 0 \leq c < m; \\
& \mathtt{m0}(m) \rightarrow \mathtt{S2}(c) : 0 \leq c < m; \\
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$$B^{\mathtt{m0}} = \{ (\mathtt{m0}(m) \rightarrow \mathtt{S1}(c)) \rightarrow \mathtt{m1}(c);$$
$$(\mathtt{m0}(m) \rightarrow \mathtt{S2}(c)) \rightarrow \mathtt{m2}(2m - c) \}$$
$$B^{\mathtt{m1}} = \{ (\mathtt{m1}(k) \rightarrow \mathtt{S4}(i)) \rightarrow \mathtt{m2}(i) \}$$

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$$\mathrm{cap}_{\mathtt{m1}}(k) = \sum_{1 \le i \le k} (1 + \mathrm{ret}_{\mathtt{m2}}(i))$$

$$\mathrm{ret}_{\mathtt{m1}} = \{\, \mathtt{m1}(k) \to 0 \,\}$$

$$\mathrm{cap}_{\mathtt{m1}} = \left(\{\, (\mathtt{m1}(k) \to \mathtt{S3}(i)) \to 1 \,\} + \mathrm{ret}_{\mathtt{m2}} \circ B^{\mathtt{m1}}\right) \circ M$$

$$\mathrm{memRq}_{\mathtt{m1}} = \mathrm{cap}_{\mathtt{m1}} + \left(\mathrm{memRq}_{\mathtt{m2}} \circ B^{\mathtt{m1}} \circ M\right)$$

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$$= \left\{ \mathtt{m1}(k) \to \frac{3}{2}k + \frac{1}{2}k^2 : k \ge 1 \right\}$$

$$\mathrm{memRq}_{\mathtt{m1}} = \mathrm{cap}_{\mathtt{m1}} + \left(\mathrm{memRq}_{\mathtt{m2}} \circ B^{\mathtt{m1}} \circ M\right)$$

$$= \left\{ \mathtt{m1}(k) \to \max\left(\frac{5}{2}k + \frac{1}{2}k^2\right) : k \ge 1 \right\}$$

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Dynamic Memory Requirement Estimation

```
void m0(int m) {
    for (c = 0; c < m; c++) {
        m1(c);                              /* S1 */
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}
```

$$B^{\mathtt{m0}} = \{\, (\mathtt{m0}(k) \to \mathtt{S1}(c)) \to \mathtt{m1}(c); (\mathtt{m0}(k) \to \mathtt{S2}(c)) \to \mathtt{m2}(2m - c) \,\}$$

$$\mathrm{ret}_{\mathtt{m0}} = \{\, \mathtt{m0}(m) \to 0 \,\}$$

$$\mathrm{cap}_{\mathtt{m0}} = (\mathrm{ret}_{\mathtt{m1}} + \mathrm{ret}_{\mathtt{m2}}) \circ B^{\mathtt{m0}} \circ M$$

$$\mathrm{memRq}_{\mathtt{m0}} = \mathrm{cap}_{\mathtt{m0}} + \left((\mathrm{memRq}_{\mathtt{m1}} + \mathrm{memRq}_{\mathtt{m2}}) \circ B^{\mathtt{m0}} \circ M\right)$$

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$$= \left\{ \mathtt{m0}(m) \to \frac{1}{2}m + \frac{3}{2}m^2 : m \geq 1 \right\}$$

$$\mathrm{memRq}_{\mathtt{m0}} = \mathrm{cap}_{\mathtt{m0}} + ((\mathrm{memRq}_{\mathtt{m1}} + \mathrm{memRq}_{\mathtt{m2}}) \circ B^{\mathtt{m0}} \circ M)$$

$$= \left\{ \mathtt{m0}(m) \to \max\left(-2 + 2m + 2m^2, \frac{5}{2}m + \frac{3}{2}m^2\right) : m \geq 1 \right\}$$

Outline

Positive Powers

Definition (Power of a Relation)

Let R be a Presburger relation and k a positive integer, then power k of relation R is defined as

$$R^k := \begin{cases} R & \text{if } k = 1 \\ R \circ R^{k-1} & \text{if } k \geq 2. \end{cases}$$

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Example

$$R = \{ (x) \rightarrow (x+1) \}$$
$$R^k = \{ (x) \rightarrow (x+k) : k \geq 1 \}$$

Transitive Closures

Definition (Transitive Closure of a Relation)

Let R be a Presburger relation, then the transitive closure R^+ of R is the union of all positive powers of R,

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Definition (Transitive Closure of a Relation, Alternative)

Inductive definition:

$$R^+ := R \cup \left(R \circ R^+ \right)$$

Transitive Closures — Approximation

Fact

Given a Presburger relation R, the power R^k (with k a parameter) and the transitive closure R^+ may not be Presburger relations.

Example

$$R = \{ (x) \to (2\,x) \}$$
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Transitive Closures — Approximation

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Given a Presburger relation R, the power R^k (with k a parameter) and the transitive closure R^+ may not be Presburger relations.

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$$R = \{ (x) \to (2\,x) \}$$
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$\Rightarrow$ need for approximation

- overapproximation $R^{\overline{+}}$
- underapproximation $R^{\underline{+}}$

[15, 22]

Transitive Closures — Approximation

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Note

Do not use transitive closures if there is an alternative.

Part II

Tools

Outline

Availability — Representation

[20]

$$\{ A(i) : 0 \leq i \leq n; B(i,j) : \exists \alpha : i = 2\alpha \}$$

- Named (and nested) spaces: `isl`

```
[n] -> { A[i]: 0 <= i <= n; B[i,j]: exists a: i = 2 a }
```

In `omega(+)`:

Availability — Representation

[20]

$$\{ A(i) : 0 \leq i \leq n; B(i,j) : \exists \alpha : i = 2\alpha \}$$

- Named (and nested) spaces: `isl`

```
[n] -> { A[i]: 0 <= i <= n; B[i,j]: exists a: i = 2 a }
```

In omega(+):

A

padding

B

```
symbolic n;
{ [0, i, 0]: 0 <= i <= n } union { [1, i,j]: exists a: i = 2 a }
```

Availability — Representation

[20]

$$\{ A(i) : 0 \leq i \leq n; B(i,j) : \exists \alpha : i = 2\alpha \}$$

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[n] -> { A[i]: 0 <= i <= n; B[i,j]: exists a: i = 2 a }
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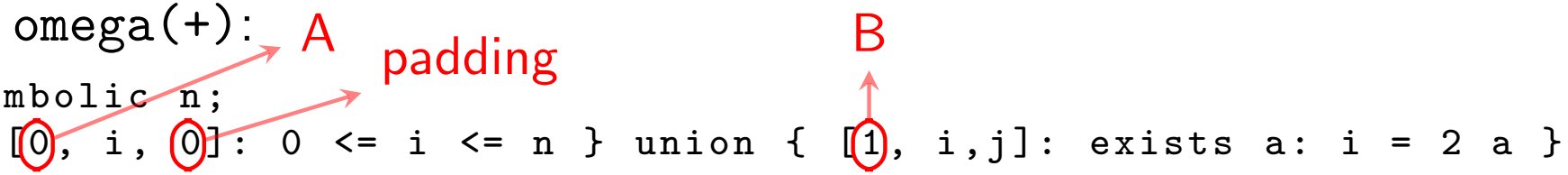
In `omega(+)`:



- Presburger sets and relations: `isl`, `omega(+)`

Availability — Representation

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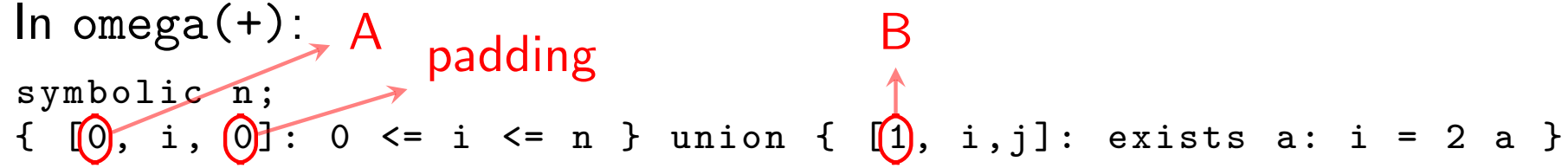


- Presburger sets and relations: `isl`, `omega(+)`

In `PolyLib`:

Moreover: `PolyLib` deals with rational sets (polyhedra)

Availability — Representation (2)

- Uninterpreted functions: `omega(+)`
  - $\Rightarrow$ arity can be greater than 0
  - $\Rightarrow$ not available in `isl` (yet)

  Note: support in `omega(+)` for uninterpreted functions is very restrictive
  - arguments need to be prefix of input/output dimensions
    - $\Rightarrow$ essentially symbolic constants

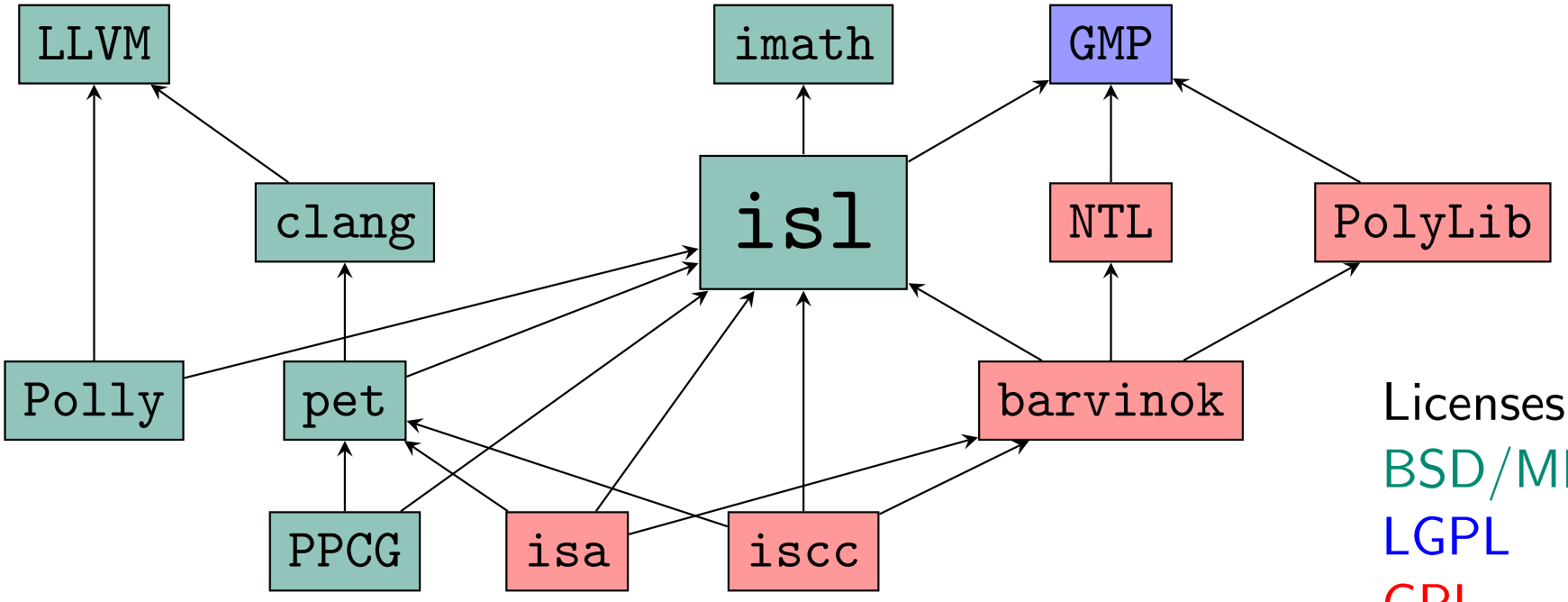
Availability — Operations

	union	intersection	difference	emptiness	domain range inverse application
`PolyLib`	✓	✓	in $\mathbb{Z}$		
`PPL`	✓	✓		✓	
`PIP`				lexmin	
`omega(+)`	✓	✓	✓	✓	✓
`isl`	✓	✓	✓	✓	✓
`barvinok`				card	
`bernstein`					
`LattE`				card	

Availability — Operations (2)

	lexmin	cardinality	bounds on polynomials	weighted counting	transitive closure
`PolyLib`		✓			
`PPL`	✓				
`PIP`	✓				
`omega(+)`	Presburger				"$R^{\pm}$"
`isl`	✓		✓		R^{+}
`barvinok`	partial	✓		✓	
`bernstein`			✓		
`LattE`		✓		✓	

isl and Related Libraries and Tools



isl: manipulates parametric affine sets and relations

barvinok: counts elements in parametric affine sets and relations

pet: extracts polyhedral model from clang AST

PPCG: Polyhedral Parallel Code Generator

iscc: interactive calculator

isa: prototype tool set including derivation of process networks and equivalence checker

Overview of `isl`

`isl` is a thread-safe C library for manipulating **integer sets and relations**

- bounded by *affine constraints*
- involving *symbolic constants* and
- *existentially quantified variables*

and **quasi-affine** and **quasi-polynomial functions** on such domains

Supported operations by core library include

- *intersection*
- *union*
- *set difference*
- *integer projection*
- *coalescing*
- *closed convex hull*
- *sampling*, *scanning*
- *integer affine hull*
- *lexicographic optimization*
- *transitive closure* (approx.)
- *parametric vertex enumeration*
- *bounds on quasipolynomials*

Polyhedral compilation library

- *schedule trees*
- *dataflow analysis*
- *scheduling*
- *AST generation*

PPCG

PPCG (http://ppcg.gforge.inria.fr/)

- Input: C code
- Output: CUDA or OpenCL code for GPGPUs

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- Input: C code
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Steps:

- extract polyhedral model from C code (pet)
- dependence analysis (isl)
- scheduling
  - expose parallelism and tiling opportunities (isl)
  - perform tiling (isl)
  - separate into parts mapped to host, GPU blocks and GPU threads
- memory management
  - add transfers of data to/from GPU
  - detect array reference groups
  - allocate groups to registers and shared memory
- generate AST (isl)

PPCG Example — Input

```
void matmul(int M, int N, int K,
            float A[static const restrict M][K],
            float B[static const restrict K][N],
            float C[static const restrict M][N])
{
  for (int i = 0; i < M; i++)
    for (int j = 0; j < N; j++) {
S1:   C[i][j] = 0;
      for (int k = 0; k < K; k++)
S2:     C[i][j] = C[i][j] + A[i][k] * B[k][j];
    }
}
```

Options:

```
--ctx="[M,N,K] -> { : M = N = K = 256 }"
--sizes="{ kernel[i] -> tile[16,16,16];
           kernel[i] -> block[8,16] }"
--pet-autodetect
```

PPCG Example — Output

```
long b0 = blockIdx.y, b1 = blockIdx.x;
long t0 = threadIdx.y, t1 = threadIdx.x;
__shared__ float s_A[16][16];
float p_C[2][1];
__shared__ float s_B[16][16];

for (long g9 = 0; g9 <= 15; g9 += 1) {
  for (long c0 = t0; c0 <= 15; c0 += 8)
    s_B[c0][t1] = B[(16 * g9 + c0) * (256) + 16 * b1 + t1];
  for (long c0 = t0; c0 <= 15; c0 += 8)
    s_A[c0][t1] = A[(16 * b0 + c0) * (256) + t1 + 16 * g9];
  __syncthreads();
  if (g9 == 0) {
    p_C[0][0] = (0);
    p_C[1][0] = (0);
  }
  for (long c3 = 0; c3 <= 15; c3 += 1) {
    p_C[0][0] = (p_C[0][0] + (s_A[t0][c3] * s_B[c3][t1]));
    p_C[1][0] = (p_C[1][0] + (s_A[t0 + 8][c3] * s_B[c3][t1]));
  }
  __syncthreads();
}
C[(16 * b0 + t0) * (256) + 16 * b1 + t1] = p_C[0][0];
C[(16 * b0 + t0 + 8) * (256) + 16 * b1 + t1] = p_C[1][0];
```

CARP Project

Design tools and techniques to aid **Correct and Efficient Accelerator Programming**

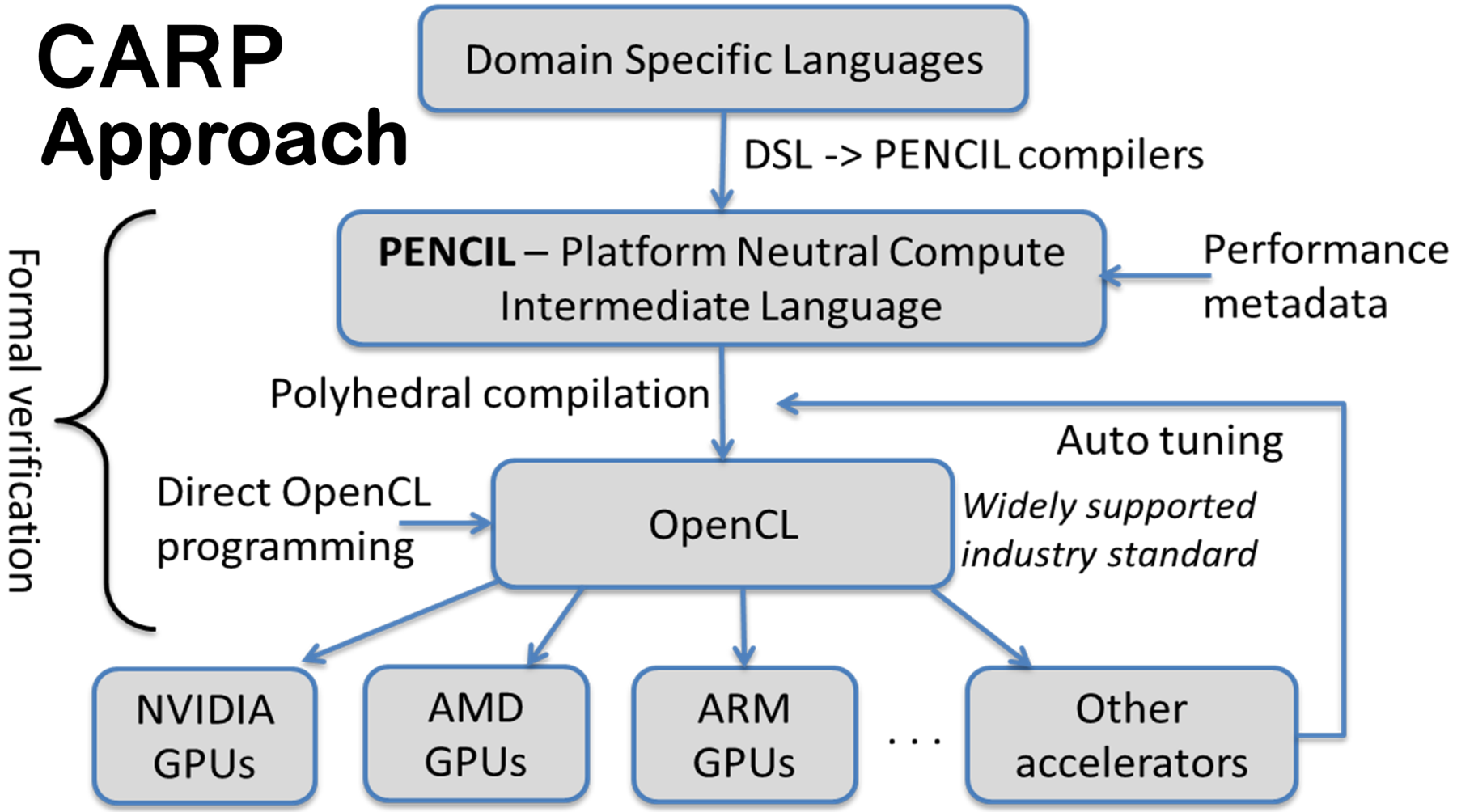
Key areas:

- High level programming models
- Advanced compilation techniques
- Formal verification

Partners:

- Imperial College London (UK)
- ENS (FR)
- ARM (UK)
- Realeyes (ES)
- RWTH Aachen University (DE)
- Monoidics (UK)
- University of Twente (NL)
- Rightware (FI)

CARP Approach



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